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Financial Cost Allocations: A Game-Theoretic Approach

Jeffrey L. Callen

ABSTRACT: Arthur L. Thomas has argued that financial cost allocations in general and depreciation allocations in particular are arbitrary and incorrigible whenever the firm's revenues are generated by interacting assets. The game-theoretic Shapley technique is applied to the net-revenue-contributions approach to depreciation allocations. The resulting allocations, it is maintained, are non-arbitrary and corrigible if statement users and the accounting profession are willing to accept a constitution of three "reasonable" allocation axioms.

In two research studies published by the American Accounting Association, Arthur L. Thomas [1969, 1974] concludes that financial cost allocations are not only arbitrary but also incorrigible, *i.e.*, incapable of verification or refutation by reference to external "real world" phenomena. While maintaining that the logic can be generalized to all financial cost allocations, Thomas particularizes his argument to depreciation allocations. He contends that either depreciation allocations are essentially arbitrary because they have no theoretical justification, or they are predicated on a net-revenue-contributions (NRC) approach. The latter appears to be justifiable since the resulting allocation pattern follows the expected net revenue contributions of the asset or project to the firm-entity. However, even NRC allocations cannot be justified if the inputs to the revenue generating process interact to produce the revenues of the firm. Unless the inputs operate independently of each other, the allocation of depreciation over time must result in the arbitrary allocation of a joint cost to a specific asset. Since there is no unique and identifiable cause-and-effect relationship between a specific asset and the revenues generated by interacting assets, any and all allocations are equally justifiable and, therefore, incorrigible.[1]

In this paper I hope to demonstrate that, time-dependent financial cost allocations, such as depreciation, need not be arbitrary or incorrigible even though asset interactions are prevalent. In what follows, the first section illustrates Thomas's argument in a simple depreciation allocation example. The next section briefly reviews the Shapley technique for allocating joint costs and applies it to my example. The final section concludes that Shapley values represent a defensible and corrigible cost allocation mechanism.

The Cost Allocation Problem, An Example

Consider the three sets of cashflows in

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[1] These conclusions have not been universally accepted. See Eckel [1976] and Thomas [1978].

Jeffrey L. Callen is Assistant Professor of Finance and Business Economics, McMaster University.

Table 1 which are assumed to eminate from the hypothetical projects *A*, *B* and *C*. Without loss of generality, each project is assumed to have a three-year life and no salvage value at the end of the period. The internal rates of return of the projects are shown under the cashflows. The cashflows for projects *A* and *B* assume one or the other project is undertaken but not both. On the other hand, project *C* represents the simultaneous investment in *A* and *B*. Also, *C*'s cashflows are set to reflect the synergistic benefits of project interaction in that *C*'s revenues are greater than and cost less than the sum of the component revenues and costs.

The allocation problem can be illustrated with reference to Table 2 which lists in columns (1) through (3), the NRC depreciation schedules for projects *A*, *B* and *C*, respectively. Project interaction evidently precludes allocating *C*'s depreciation schedule on the basis of the independent schedules. Columns (1) and (2) of Table 2 do not sum to column (3).

Another seemingly reasonable approach might be to allocate by incremental depreciation charges. The incremental charges allocated to project *A*, column (4) of Table 2, are obtained by subtracting column (2) from (3). Similarly, the incremental depreciation schedule for project *B* is the difference between columns (3) and (1). Which, however, is the incremental project? If *A* is assumed to be incremental to *B*, then column (4) would be allocated to *A* and column (2) to *B*. On the other hand, if *B* is incremental to *A*, then column (5) is allocated to *B* and column (1) to *A*. Accepting one or the other makes the arbitrary ordering of the projects fundamental to the allocation process and, therefore, the allocation is incorrigible. One could just as reasonably allocate one half of *C*'s depreciation schedule to each of the component projects.

TABLE 1
COSTS, NET REVENUES AND INTERNAL RATES OF RETURN FOR PROJECTS *A*, *B* AND *C*

Year	*Project A*	*Project B*	*Project C*
0	−$1,000	−$500	−$1,484
1	278	94	450
2	450	250	750
3	500	250	800
IRR	10%	8%	15%

It is worth noting that the allocation problem does not depend on accepting simultaneous interacting projects. Any time the firm invests in a new asset, there are bound to be interactions between the firm's capacity to generate revenues and the new asset. Few projects are likely to give the same cashflows when divorced from the remainder of the firm's assets. The question again arises: how much of the firm's revenues should be allocated to the new project and how much to existing assets? What proportion of the firm's depreciation schedule should be allocated to the new project and what proportion to existing assets?

Shapley Values

The application of the Shapley tech-

TABLE 2
NET REVENUE CONTRIBUTION DEPRECIATION SCHEDULES FOR PROJECTS *A*, *B* AND *C* AND INCREMENTAL DEPRECIATION SCHEDULES FOR PROJECTS *A* AND *B*

NRC Depreciation Schedules				*Incremental Depreciation Schedules*	
Year	*(1) Project A*	*(2) Project B*	*(3) Project C*	*(4) Project A*	*(5) Project B*
1	$178	$ 54	$227	$173	$ 49
2	367	215	561	346	194
3	455	231	696	465	241

nique to accounting methodology was first advocated by Shubik [1962] in an interesting but, until recently, neglected paper.[2] Shubik illustrated the utility of the Shapley approach for allocating joint costs in transfer pricing problems.[3] Also working in a static transfer pricing framework, Hamlen, Hamlen and Tschirhart [1977] found a number of cost allocation procedures, including Shapley values, to be useful potentially for decentralized decision-making in decreasing cost firms.[4] If we are to appreciate the applicability of Shapley values to the NRC method for allocating depreciation, a brief introduction to some elementary game-theoretic notions is in order.

The theory of games conceptualizes a measure of interaction between players in a joint venture or coalition in comparison to their effectiveness as individuals. This measure of the effectiveness of cooperation is called the characteristic function of the game. The characteristic function (V) is defined over all potential coalitions and is assumed to be superadditive. This means that value of the players acting independently cannot be greater than their value in a coalition. In a two person coalition superadditivity is defined by the mathematical relationship

$$V(A,B) \geq V(A) + V(B)$$

where $V(A)$, $V(B)$ and $V(A,B)$ are the characteristic functions of player A, player B and the coalition comprised of A and B, respectively.

Shapley values allocate the benefits of the coalition to each player in a specific and unique manner. Each player is valued by his incremental benefit to the coalition. Since the incremental benefit is not invariant to the order in which the player is presumed to join the coalition, each possible alternative is assumed to be equi-probable and weighted accordingly. For example, in a two player coalition, the incremental value of player A is $V(A)$ if A enters the coalition first and $V(A,B) - V(B)$ if B is first. Assuming each occurrence is equally likely, the allocation to player A is

$$S_A = 1/2\, V(A) + 1/2\,[V(A,B) - V(B)]$$

Similarly, B's allocation is

$$S_B = 1/2\, V(B) + 1/2\,[V(A,B) - V(A)]$$

Total coalition benefits are allocated by this technique since

$$S_A + S_B = V(A,B)$$

In a three player coalition, the concept is the same but the allocation formula is more complex. If A, B and C are the players then the relevant characteristic functions for all possible coalitions are:

$$V(A), V(B), V(C)\ V(A,B), V(A,C), V(B,C) \quad \text{and} \quad V(A,B,C).$$

The Shapley value allocations would be:

$$\begin{aligned} S_A &= 1/3\, V(A) + 1/6\,[V(A,B) - V(B)] \\ &\quad + 1/6\,[V(A,C) - V(C)] \\ &\quad + 1/3\,[V(A,B,C) - V(B,C)] \\ S_B &= 1/3\, V(B) + 1/6\,[V(A,B) - V(A)] \\ &\quad + 1/6\,[V(B,C) - V(C)] \\ &\quad + 1/3\,[V(A,B,C) - V(A,C)] \\ S_C &= 1/3\, V(C) + 1/6\,[V(A,C) - V(A)] \\ &\quad + 1/6\,[V(B,C) - V(B)] \\ &\quad + 1/3\,[V(A,B,C) - V(A,B)] \end{aligned}$$

Again, all coalition benefits are allocated since

$$S_A + S_B + S_C = V(A,B,C)$$

[2] Shapley values were first introduced in their natural game theory setting by Shapley [1953].

[3] Others have applied the Shapley technique to merger benefits and public utility pricing schemes. See Mossin [1968], Littlechild [1970], and Loehman and Winston [1971, 1974].

[4] In a similar setting, Jensen [1972] analyzes the Shapley approach for allocating the cost of a shared facility under different cost function specifications.

TABLE 3
SHAPLEY VALUE ALLOCATED CASHFLOWS, INTERNAL RATES OF RETURN, AND SHAPLEY VALUE DEPRECIATION SCHEDULES FOR PROJECTS *A* AND *B*

	Shapley Value Cashflow		*Shapley Value Depreciation Schedules*	
Year	(1) *Project A*	(2) *Project B*	(3) *Project A*	(4) *Project B*
0	−$992	−$492	—	—
1	317	133	$175	$ 52
2	475	275	357	204
3	525	275	460	236
IRR	14.3%	16.3%		

Shapley values can be generalized to an n player coalition. The value of the jth player in an n player coalition is

$$S_j = \sum_{G \subset J} \frac{(n-g)!(g-1)!}{n!} \cdot [V(G) - V(G - \{j\})]$$

where $J = \{j: j = 1, \cdots, n\}$ is the set of players and G is any subset (coalition) of g players. The incremental benefit conferred on the coalition by player j when he is in the gth position is weighted by the term

$$\frac{(n-g)!(g-1)!}{n!}.$$

$n!$ is the total number of possible coalitions while $(g-1)!$ and $(n-g)!$ represent the number of ways of ordering the players in coalition G and $J-G$, respectively.

If for players and coalitions we substitute the terms projects and firms, the transition to our immediate interest is obvious. Define the characteristic function of a project to be the schedule of cashflows which result from the project. This function is superadditive provided, for each corresponding year, the cashflow from interacting projects is greater than or equal to the sum of cashflows of the separate projects. The cashflows in our example are superadditive.

Shapley values can be applied to our example in one of two equivalent ways. Either the cashflows are allocated initially and then the NRC depreciation schedules are calculated, or the depreciation schedules are determined first and subsequently allocated to the component projects. The first approach is illustrated in Table 3. Columns (1) and (2) are the Shapley value allocated cashflows for projects *A* and *B*, respectively. The internal rates of return are calculated for these cashflows and then the NRC depreciation schedules. Column (3) gives *A*'s depreciation schedule and column (4) that of *B*'s. Equivalently the Shapley value technique can be applied directly to the depreciation schedules in Table 2. For example, *A*'s depreciation schedule [Column (3), Table 3] can be determined by multiplying each of columns (1) and (4) of Table 2 by .5 and adding the result.[5]

The Corrigibility of Shapley Values

Specifying an additional cost allocation procedure, albeit one which takes account of project interaction, does not of itself alleviate the incorrigibility problem. It is not the paucity of allocation procedures which is problematic but the

[5] So that the Shapley approach splits the interaction effect evenly between the two projects.

reverse. If the Shapley value approach is valid, it must be defensible against all other cost allocation procedures.

There are two criteria by which a cost allocation procedure can be evaluated. One criterion requires that allocation avoid leading to dysfunctional decisions. For example, in a transfer pricing context, Hamlen, Hamlen and Tschirhart reject those allocations which result in divisional decisions which are suboptimal at the corporate level. This criterion, which they call neutrality, is far less operational than it appears. Any number of joint cost allocation procedures may be neutral with respect to a specific decision.[6] A more fundamental problem is that an allocation procedure may be neutral with respect to one decision but not another. This problem is further compounded in financial accounting allocations where the decision itself is a function of the external user and neutrality is difficult, if not impossible, to define. Therefore, the neutrality criterion is unlikely to differentiate among allocation techniques except where the accounting system is oriented to a specific decision.

Instead, I would maintain that an alternative criterion is operable and should guide the acceptability of a joint cost allocation procedure. Does the procedure satisfy a "reasonable" set of cost allocation axioms with respect to which it alone is optimal? Provided these axioms are perceived to be "reasonable" to those concerned, statement users and accountants, the allocation technique can be defended against rival approaches.

Mossin [1968] has shown three axioms to be necessary and sufficient for the optimality of the Shapley value approach. If these axioms comprise an acceptable cost allocation constitution, then Shapley values are the only valid cost allocation procedure. Although these axioms can be described in general terms, we will state them in the context of depreciation allocations.

Axiom 1. If the firm invests in a noninteractive project, the projects depreciation schedule is a function of its own revenue-cost structure and independent of the firm.

Axiom 2. If the firm invests in two projects which do not interact with each other (although each may interact with the firm), the depreciation schedule allocated to the projects simultaneously is equal to the sum of the depreciation schedules allocated to the separate projects.

Axiom 3. Projects with the same revenue-cost structure are allocated the same depreciation schedules.

Conclusion

Although Thomas argues in his research studies that much of financial accounting is void of meaningful content, his conclusions are somewhat premature. Using the same cost allocation backdrop as does Thomas, namely, depreciation, I have shown that cost allocations need not be arbitrary or incorrigible As long as statement users and the accounting profession are willing to accept (1) a constitution of three simple cost allocation axioms, and (2) the necessarily concomitant Shapley value allocation procedure, Thomas paints too gloomy a picture. It would be unfortunate should the accounting profession view the potential application of game theory to accounting to be too esoteric. In the absence of a uniquely justifiable cost allocation scheme, such as Shapley values, Thomas is probably correct.

[6] Employing the theory of the core, Hamlen, Hamlen and Tschirhart [1977] found two allocation techniques besides Shapley values which discourage globally suboptimal divisional decisions.

REFERENCES

Eckel, L. G., "Arbitrary and Incorrigible Allocations," THE ACCOUNTING REVIEW (October 1976), pp. 764–777.

Hamlen, S. S., W. A. Hamlen, Jr. and J. T. Tschirhart, "The Use of Core Theory in Evaluating Joint Cost Allocation Schemes," THE ACCOUNTING REVIEW (July 1977), pp. 616–627.

Jensen, D. L., "A Class of Mutually Satisfactory Allocations," THE ACCOUNTING REVIEW, (October, 1977), pp. 842–856.

Littlechild, S. C., "A Game-Theoretic Approach to Public Utility Pricing," *Western Economic Journal* (June 1970), pp. 162–166.

Loehman, E. T., and A. B. Whinston, "A New Theory of Pricing and Decision Making for Public Investment," *Bell Journal of Economics and Management Science* (Autumn 1971), pp. 606–625.

———, "An Axiomatic Approach to Cost Allocations for Public Investment," *Public Finance Quarterly* (April 1974), pp. 236–251.

Mossin, J., "Merger Agreements: Some Game-Theoretic Considerations," *Journal of Business* (October 1968), pp. 460–471.

Shapley, L. S., "The Value of an *N*-Person Game," in H. W. Kuhn and A. W. Tucker, eds., *Contributions to the Theory Games, Vol. 2, Annals of Mathematical Studies No. 28* (Princeton University Press, 1953), pp. 307–317.

Shubik, M., "Incentives, Decentralized Control, The Assignment of Joint Costs and Internal Pricing," *Management Science* (April 1962), pp. 325–343.

Thomas, A. L., "The Allocation Problem in Financial Accounting Theory," *Studies in Accounting Research No. 3* (American Accounting Association, 1969).

———, "The Allocation Problem: Part Two," *Studies in Accounting Research No. 9* (American Accounting Association, 1974).

———, "Arbitrary and Incorrigible Allocations: A Comment," THE ACCOUNTING REVIEW, (January 1978), pp. 263–269.