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Large time and small noise asymptotic results for mean reverting diffusion processes with applications*

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Summary. We use the theory of large deviations to investigate the large time behavior and the small noise asymptotics of random economic processes whose evolutions are governed by mean-reverting stochastic differential equations with (i) constant and (ii) state dependent noise terms. We explicitly show that the probability is exponentially small that the time averages of these process will occupy regions distinct from their stable equilibrium position. We also demonstrate that as the noise parameter decreases, there is an exponential convergence to the stable position. Applications of large deviation techniques and public policy implications of our results for regulators are explored.

Keywords and Phrases: Large deviations, Level-2-large deviations, Exit problems, Mean reverting stochastic differential equations.

JEL Classification Numbers: C00, G10.

1 Introduction

This paper uses the theory of large deviations to investigate both the large time behavior and the small noise asymptotics of random economic processes whose evolutions are governed by mean-reverting stochastic differential equations with (i) constant and (ii) state dependent noise terms. It is to be expected, that in

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the absence of noise, these processes will converge (or mean revert) to a stable equilibrium, at least over the long run. However, in the presence of noise, of even small magnitudes, convergence may not obtain over large times.

A question naturally arises: if only the time series realization of the process is observed, does the time average provide any information about the equilibrium value? For both processes, we obtain sharp estimates for the large time behavior of the time averages and the probabilistic characterization of their deviation from the stable equilibrium. The empirical measures are shown to converge to a distribution and the time average to the equilibrium value. Explicit solutions are provided for the probability that the average will spend time in regions that are different from the equilibrium value. In particular, it is shown that this average will assume values in regions distinctly different from the equilibrium value with exponentially small (but positive) probabilities. This may prevent the observer of the time series from correctly estimating the true value even over large times. Policy implications of this result are discussed in the conclusion.

In addition to these large time results, the small noise asymptotics for both types of processes are investigated. In particular, we analyze the speed with which convergence to fundamental value takes place as noise reduces to zero. Explicit solutions are provided. Specifically, it is shown that convergence to fundamental value is exponentially fast as noise is eliminated. These results provide interesting insights regarding the role of noise, convergence and the long term behavior of models based on such stochastic processes. The policy implications of these results are also explored.

As mentioned above, both issues addressed in this paper are analyzed using tools from the mathematical *theory of large deviations.*[1] Specifically, the large time asymptotics draw on techniques from what is referred to as the theory of *level-2 large deviations.* The small noise asymptotics borrow techniques from the related theory of Friedlin and Wentzell (1984), which they apply to a class related theory of Friedlin and Wentzell (1984), which they apply to a class of problems called "*exit problems*".[2] We provide a brief introduction to the theory of large deviations in Appendix A. Details of the level-2 large deviation techniques used in this paper (with appropriate references) are given in Appendix B. Details of the Freidlin-Wentzell techniques are found in Appendix C.[3]

There are many potential applications of our results to diverse areas of economics. Furthermore, the techniques from the theory of large deviations used in

[1] One could try to derive these results from first principles or some other method. But alternative techniques would most likely be cumbersome and tedious. We find that tools from the established theory of Large Deviations are relatively easy to apply.

[2] The term "*exit problem*" is easily understood in the context of our paper. We are interested in the rate of convergence of economic (diffusion type) processes to equilibrium trajectories as the stochastic disturbance term dies down. In other words, we are concerned with the rate at which the probability of "*wild*" deviations into regions far from the equilibrium path goes to zero. Equivalently, we are computing the rate at which the probability of "*exiting*" into regions away from equilibrium reduces to zero. These kind of problems were first studied by Friedlin and Wentzell (1984).

[3] A more comprehensive discussion of the theory of large deviations can be found in Deuschel and Stroock (1989).

arriving at our results are likely to be of interest to economic theorists generally. In financial economics, for example, there is persuasive empirical evidence that financial markets may not be fully efficient in the traditional sense of Fama (1965) or Samuelson (1965) (see, for example, Bernard and Thomas, 1990; Fama and French, 1992). In fact, there is a considerable amount of empirical work supporting the hypothesis that prices do not follow a martingale process with positive drift but are mean reverting instead (see Cutler, Poterba, and Summers, 1989; Fama and French, 1988; Lo and Wang, 1995; Poterba and Summers, 1988, for example). Many believe that the reasons for such inefficiencies, predictability and mean reversion are the trading activities of liquidity "noise" traders. These are uninformed traders who are assumed to participate to a significant degree in financial markets and who are assumed to trade for liquidity reasons alone (see Kyle, 1985; Black, 1986; De Long et al., 1990a, 1991). The early arguments by Friedman (1953) and Fama (1965) that (i) noise traders do not carry weight in price formation and (ii) even if they did, they could not last for any significant length of time trading with market arbitrageurs, were cited until recently as reasons for "ignoring" liquidity noise trading in asset pricing models. These arguments have largely been rejected. The recent thinking on this issue is that arbitrageurs who are risk averse or have shorter term horizons, may take only limited positions in the market when faced with liquidity noise traders who are bidding prices way up (or down) from "fundamental" values[4], especially if there is a risk that prices may take a very long time to revert to their fundamental value.[5] In sum, there seems to be a growing acceptance in the financial economics literature that prices in financial markets are mean reverting, but can significantly deviate from their fundamental values over time.[6] Our large time asymptotic results can be used directly to study the probability of deviations of the time average from fundamental value in order to obtain quantitative and qualitative insights into noise induced "inefficiency" in the sense of Black (1986). As we discuss later in the concluding section, this issue is of importance not only to the research community but also to policy makers.

[4] Fundamental value is typically understood to be the value assessed by the arbitrageurs (equivalently, persons with superior knowledge or insiders).

[5] Shiller (1984) and Cambell and Kyle (1987) showed that risk aversion among arbitrageurs, even with infinite horizons, is sufficient to dampen arbitrage. DeLong et al. (1990a) maintain that "noise itself creates risk" and that "noise traders create their own space". More specifically, they argue that there is an additional source of risk faced by arbitrageurs with short horizons, namely, that noise traders' "beliefs" may not revert to the mean for a very long time and, in the short run, asset prices may tend to exhibit large deviations from fundamental values. Perhaps, the essence of this discussion is best summarized by Black (1986), when he defines an efficient market "as one in which price is within a factor of two of value...the factor of two is arbitrary..."

[6] Follmer and Schweizer (1993) develop a general model of asset pricing that supports the mean reverting diffusion process for security prices in a market populated by (i) arbitrageurs who have knowledge regarding the fundamental or intrinsic value of the security, (ii) uninformed liquidity traders who trade for liquidity reasons and (iii) noise traders who are uninformed and trade based on technical analysis. Their model of security prices allows for multiple sources of parametric uncertainties in addition to liquidity noise. Eliminating these other sources of uncertainties, their model yields a mean reverting diffusion model for the (log of) security prices with liquidity noise capture by a one dimensional Brownian motion. The liquidity noise can be state (price) dependent or independent.

The small noise asymptotics computed by us give the rate of convergence of price to fundamental value as liquidity noise dies out. This is an important economic issue since liquidity noise sustains wasteful trade. Eliminating this noise yields a fully informed economy and reduces the deadweight losses of wasteful trading, thereby improving social welfare. It is comforting to know that the convergence to fundamental value takes place exponentially fast as liquidity noise is eliminated. To the extent that liquidity noise trading is believed to be welfare reducing, our results are of potential importance to regulators and policy planners concerned with the capital markets such as the Securities and Exchange Commission (SEC) and the Financial Accounting Standards Board (FASB). We briefly explore the implications of our analysis for the information disclosure policies of these bodies in the last section.

Other interesting examples where the asymptotic issues studied here are of relevance include: (a) the study and characterization of the probabilistic behavior of interest rate processes that exhibit mean reversion such as the model proposed by Vasicek (1977), or the model by Cox, Ingersoll and Ross (1985); (b) the derivation of bounds for derivative prices, such as options written on interest rates or any other asset whose prices are mean reverting. (The computation of these bounds depend explicitly on the probabilistic behavior of the prices of the underlying primitives (see Grundy, 1991, for example); (c) the study of the behavior of monetary markets and inflationary behavior in the presence of random monetary shocks in economic frameworks such as the one proposed by Cagan and others (as referenced in Turnovsky, 1995), and (d) the solving for equilibria in some recent models of dynamic evolution and game theory. In these evolutionary game theory models, it becomes crucial to study the asymptotic properties of evolutionary equations that are subject to random shocks in order to obtain the evolutionary stable equilibrium. These evolutionary equations are typically mean-reverting in nature. While the explicit asymptotic computations carried out in this paper are of significance to this area of research generally, they are also directly relevant to specific continuous time models of economic evolution such as the ones by Fudenberg and Harris (1992) and Foster and Young (1990).

2 Asymptotic results for mean-reverting diffusions with state independent (constant) noise

Consider the mean reverting diffusion process governing the random evolution of $r_\varepsilon(t)$[7] with a constant mean reverting pull, κ; constant θ, representing the fundamental or stable equilibrium point, and constant variance ε:[8]

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[7] The subscript in $r_\varepsilon(t)$ emphasizes that the process is stochastic in nature with noise intensity ε.

[8] Within the context of the model by Follmer and Schweizer (1993), this stochastic differential equation obtains as a special case. In their equation (3.6) on page 9 of their paper, setting

$$d\hat{Z}(t) = -\kappa dt$$

and

$$\begin{aligned} dr_\varepsilon(t) &= \kappa(\theta - r_\varepsilon(t))dt + \varepsilon dB(t) \\ r_\varepsilon(0) &= C + \theta \end{aligned} \tag{1}$$

where C and θ are real numbers. Equation (1) is a mean reverting process of the Ornstein-Uhlenbeck type. In the absence of the variance term, that is, setting ε to zero, the process in equation (1) becomes deterministic with a solution given by

$$r_0(t) = C\exp(-rt) + \theta \tag{2}$$

Proposition 1 stated below addresses the large time properties for the specific stochastic structure assumed in this section. More specifically, this proposition provides an explicit characterization of the probability that over a long period $\overline{r}_\varepsilon(t)$, the time average of the process, is found in neighborhoods distinct in value from the equilibrium fundamental value θ.

Proposition 1 *Let $r_\varepsilon(t)$ be the diffusion process governed by the stochastic differential equation*

$$dr_\varepsilon(t) = \kappa(\theta - r_\varepsilon(t))dt + \varepsilon dB(t)$$

$$r_\varepsilon(0) = C + \theta$$

where κ and ε are positive numbers, θ and C are real numbers, and B(t) is the standard Brownian motion. Let

$$\overline{r}_\varepsilon(t) = \frac{1}{t}\int_0^t r_\varepsilon(s)ds$$

represent the average value of $r_\varepsilon(t)$ for the period [0,t]. Let c, d and x be variables that represent values distinct from θ with $c < d$. Then the probability that $\overline{r}_\varepsilon(t)$ belongs to the neighborhood around x is exponentially small, that is

$$\lim_{t\uparrow\infty} \frac{1}{t}\log P(c \le \overline{r}_\varepsilon(t) \le d) = -\inf_{c\le x\le d} I(x) \tag{3}$$

where

$$I(x) = \frac{1}{2}\frac{\kappa^2}{\varepsilon^2}(x-\theta)^2$$

Proof. See Appendix B.

The explicit characterization of the function I in Proposition 1, also called the rate function, allows us to prove a number of intuitive comparative statics results. Consistent with our intuition, Proposition 1 says that not only is the probability exponentially small that the time average of the process occupies neighborhoods distinct in value from the equilibrium fundamental value θ, but it is relatively

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$$dZ(t) = \kappa\theta dt + \varepsilon dB(t)$$

where κ (the mean reverting parameter), θ (the stable equilibrium value), ε (the noise intensity) are positive constants, and $B(t)$ is the standard Brownian motion, yields the diffusion equation with constant noise. In the context of interest rate models, this diffusion process would correspond to the one proposed by Vasicek (1977). This type of diffusion process could also represent game-theoretic evolutionary processes that are state independent.

smaller the farther x is from θ. Furthermore, this probability decreases as the mean reverting force κ increases. Also, the probability increases in the variance ε.[9]

The next proposition provides an explicit solution to the second issue raised above, namely, the small noise asymptotics of the specific stochastic structure assumed in this section. We show that as the level of noise, as captured by ε, falls, $r_\varepsilon(t)$ converges (in probability) to the fundamental value exponentially quickly.[10] We now state the proposition.

Proposition 2 *Let $r_\varepsilon(t)$ be the diffusion process given by*

$$dr_\varepsilon(t) = \kappa(\theta - r_\varepsilon(t))dt + \varepsilon dB(t)$$

$$r_\varepsilon(0) = C + \theta$$

where κ and ε are positive numbers, θ and C are real numbers, and $B(t)$ is the standard Brownian Motion. Consider $r_0(t) = C\exp(-\kappa t) + \theta$, the solution to the corresponding deterministic equation. Then for any $\delta > 0$ and $T > 0$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log P(\sup_{0 \le t \le T} \mid r_\varepsilon(t) - r_0(t) \mid > \delta) = -I(\kappa, \delta, T)$$

$$\text{where } I(\kappa, \delta, T) = \kappa\delta^2 \frac{\exp(2\kappa T)}{\exp(2\kappa T) - 1} \tag{4}$$

Proof. See Appendix C.

The explicit characterization of the function I in Proposition 2 clearly demonstrates the exponential convergence of the process to the stable equilibrium value as uncertainty, ε, dies down. As was the case with Proposition 1, the rate function I of Proposition 2 has very intuitive properties. The following four properties can be verified formally:

a) Ceterus paribus, I decreases in T; that is to say, that the longer one waits, even with small uncertainties, ε, the probability that the process $r_\varepsilon(t)$ will exhibit values substantially different from the equilibrium value, θ, (i.e. of order δ or more) increases.

b) Ceterus paribus, I increases in δ; that is, as the magnitudes of the deviations of interest become larger, the smaller their probabilities of occurrence.

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[9] Interpreting $\overline{r}_\varepsilon(t)$ as the (log of) the security prices in the Follmer and Schweizer (1993) model, Proposition 1 allows one to compute the probability that the time average of observed values of $\overline{r}_\varepsilon(t)$, is efficient in the sense of Black (1986), that is to say, the probability that $\overline{r}_\varepsilon(t)$ occupies a neighborhood of interest distinct from the equilibrium value θ, say, around $\log 2\theta$, for given levels of ε, κ and θ.

[10] In the context of the financial markets of Follmer and Schweizer (1993), this would be interpreted to mean that convergence to a fully informed economy occurs exponentially fast as uniformed trading noise dies down. For the special situation where fully informed individuals have endowments on the Pareto frontier, a fully informed economy would simply have no trading.

c) Ceterus paribus, I increases in κ ; that is, with a higher mean reverting pull, the probability of exceeding values relatively farther away from equilibrium is lower for a given time horizon and uncertainty.

d) It is interesting to note that the rate function in Proposition 2 is independent of the choice of the equilibrium value, θ. This is explained by the fact the diffusion process governing $r_\varepsilon(t)$ has a variance independent of θ and the drift depends only on the difference $(\theta - r_\varepsilon(t))$. This result is in contrast to the stochastic process considered in the next section of this study.

3 Asymptotic results for a mean-reverting diffusions with state dependent noise

Consider the following diffusion process for $r_\varepsilon(t)$, with κ, θ and ε are positive numbers, and $B(t)$ the standard Brownian motion process:[11]

$$dr_\varepsilon(t) = \kappa(\theta - r_\varepsilon(t))dt + \varepsilon\sqrt{r_\varepsilon(t)}dB(t) \tag{5}$$

after setting the initial condition $r_\varepsilon(0) = \theta$.

Noise in this model increases with higher values of $r_\varepsilon(t)$ as is evident from the square root term in the variance of the diffusion process in equation (5). The square root term was chosen in order to prevent noise from increasing too fast, potentially precluding the possibility of any large deviation results.[12]

The next proposition parallels Proposition 1 by providing the answer to the question: For diffusion processes with noise an increasing function of the value of the process, what is the probability that the time average, $\overline{r}_\varepsilon(t)$, occupies neighborhoods distinct from the equilibrium value, θ?

Proposition 3 *Let $r_\varepsilon(t)$ be the diffusion process governed by the stochastic differential equation*

$$dr_\varepsilon(t) = \kappa(\theta - r_\varepsilon(t))dt + \varepsilon\sqrt{r_\varepsilon(t)}dB(t)$$

$$r_\varepsilon(0) = \theta$$

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[11] Once again this diffusion process can be derived as a special case of the model by Follmer and Schweizer (1993) by setting

$$d\hat{Z}(t) = -\kappa dt$$

and

$$dZ(t) = k\theta dt + \varepsilon\sqrt{r_\varepsilon(t)}dB(t)$$

in their equation (3.6). Here, the intensity of stochastic component is assumed to be proportional to the square-root of the state variable $r_\varepsilon(t)$. In the context of the Follmer-Schweizer model, this corresponds to noise and liquidity trading increasing in bull markets and falling off in bear markets. In the context of interest rate models, this diffusion process would correspond to the one proposed by Cox, Ingersoll and Ross (1985). This type of diffusion process could also represent game-theoretic evolutionary processes that are state dependent.

[12] It can be shown that for the cases where the intensity of the shocks is greater than for the square-root process, no large deviations results obtain. The intensity of the noise "overwhelms" the process, so to speak.

where κ and ε are positive numbers, θ is a positive real number and $B(t)$ is the standard Brownian motion. Let

$$\overline{r}_{\varepsilon}(t) = \frac{1}{t}\int_{o}^{t} r_{\varepsilon}(s)ds$$

represent the average value of $r_{\varepsilon}(t)$ for the period [0,t]. Let c, d and x be variables distinct from θ. Then the probability that $\overline{r}_{\varepsilon}(t)$ belongs to the neighborhood around x is exponentially small, that is

$$\lim_{t\uparrow\infty}\frac{1}{t}\log P(c < \overline{r}_{\varepsilon}(t) < d) = -\inf_{c\leq x\leq d} I(x) \tag{6}$$

$$I(x) = \frac{1}{2}\frac{\kappa^2}{\varepsilon^2}\left(\frac{\theta^2}{x} + x - 2\theta\right)$$

Proof. See Appendix B.

As was the case in Proposition 1, the probability that $\overline{r}_{\varepsilon}(t)$ occupies regions distinct from the equilibrium θ decreases as x moves farther away from θ. The probability also falls as the mean reverting force κ becomes larger and increases as the noise induced by liquidity demands, ε, increases. Once again, the explicit form of the solution in Proposition 3 lets us compute the probability that the time average of the process, $\overline{r}_{\varepsilon}(t)$, will occupy regions distinct from the stable equilibrium value, θ.

We now turn to the last proposition which addresses the small noise asymptotics for the diffusion process analyzed in this section. Once more, as in Proposition 2, we obtain an explicit characterization of the probability that $r_{\varepsilon}(t)$ converges to the equilibrium value. In particular, we show that convergence occurs exponentially quickly as noise decreases to zero. As before, this proposition emphasizes the potential importance of noise in the convergence to stable positions. We now state the proposition formally.

Proposition 4 *Let $r_{\varepsilon}(t)$ be the diffusion process given by the stochastic differential equation*

$$dr_{\varepsilon}(t) = \kappa(\theta - r_{\varepsilon}(t))dt + \varepsilon\sqrt{r_{\varepsilon}(t)}dB(t)$$

$$r_{\varepsilon}(0) = \theta$$

where κ and ε are positive numbers, θ is a positive real number and $B(t)$ is the standard Brownian motion. Then for any $\delta > 0$ and $T > 0$

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\log P\left(\sup_{0\leq t\leq T} | r_{\varepsilon}(t) - \theta | > \delta\right) = -I(\kappa,\theta,\delta,T)$$

where $I(\kappa,\theta,\delta,T) = (C + 2\kappa\sqrt{\theta C})\frac{e^{\kappa T}-1}{2\kappa} - 2\kappa\theta\log\frac{\sqrt{C}(e^{\kappa T}-1)+2\kappa\sqrt{\theta}}{2\kappa\sqrt{\theta}}$,

$$\begin{aligned} C &= C(\kappa,\theta,\delta,T) \\ &= \frac{2\kappa^2[\theta(e^{\kappa T}+1)^2 + 2e^{\kappa T}\delta - (e^{\kappa T}+1)\sqrt{(e^{\kappa T}+1)^2\theta^2 + 4e^{\kappa T}\theta\delta}]}{(e^{\kappa T}-1)^2}. \end{aligned} \tag{7}$$

Proof. See Appendix C.

The rate function I has properties similar to the rate function in Proposition 2 and with largely similar intuition. We content ourselves with stating these properties without too much discussion but noting that some tedious computation is required to establish them.

a) Ceterus paribus, I decreases in T. This is consistent with the intuition that deviations of order greater than δ from the equilibrium value are more probable over a longer horizon.

b) Ceterus paribus, I increases in δ signifying that as the magnitudes of the deviations of interest become larger, the smaller their probabilities of occurrence.

c) Ceterus paribus, I increases in κ. This is consistent with the intuition that as mean reverting force increases, larger deviations are less probable.

d) In contradistinction to the case in Proposition 2, the rate function I now decreases in θ. This can be explained by the intuition that as θ becomes larger, so does the value of $r_\varepsilon(t)$ around it and also the variance of the process. As a consequence, the larger is the probability of observing large deviations from the equilibrium.

4 Policy implications and concluding comments

If one were to interpret the stochastic differential equations in this paper as models of the (log) of stock price evolutions, as in Follmer and Schweitzer (1993), the source of noise can be attributed to liquidity noise traders. Then, the large deviation results obtained in this paper are of potential interest to policy makers such as the SEC and FASB in assessing the disproportionate effects of small noise on security prices and economic welfare. Policy makers get to observe only the realized prices that are influenced by uninformed noise trading. To infer the true price they could construct a long term price average of realized prices as Propositions 1 and 3 imply. However, these propositions also show that due to liquidity noise, deviations of this measure from the fundamental value, though comfortingly rare (exponentially small), will occur with measurable frequencies.[13] This is clearly seen from the fact that there is a positive probability that the time average of observed (log) market prices will occupy values distinct from their fundamental value. This may prevent the observer from estimating reasonably precisely the fundamental value. Eliminating these price inefficiencies by reducing liquidity noise therefore becomes important for policy makers concerned with the capital markets.

Propositions 2 and 4 imply that the reduction of liquidity noise brings the economy back to efficiency *exponentially quickly* so that prices reflect fundamental value rather than noise. To the extent that policy makers are able to reduce

[13] The problem with rare events is that they can have rather large effects on the economy as the market crash of 1987 attests. Shleifer and Summers (1990) among others have attributed the market crash to the interaction between informed traders and liquidity noise traders.

noise even a little bit, through disclosure regulations for example, it may reduce wasteful trading exponentially thereby increasing economic welfare.[14]

To conclude, this paper has analyzed two different types of financial markets, one where noise from liquidity trades is independent of the value process and the other where it depends on the value process. In both cases, it was found that the probability of occupying certain neighborhoods distinct from equilibrium is both computable and exponentially small. It was also shown that as liquidity noise decreases, convergence to a fully informed economy occurs exponentially fast. This suggests that policy planners interested in capital market operations should help promulgate liquidity noise reducing regulations. Clearly there is scope for further research on the subject of the role of noise in more complex economies and its implications for welfare and policy planning.

Appendix A

Large deviations principles

The theory of Large Deviations provides rates at which probabilities of certain events converge to zero. Such events can usually be thought of as "rare". A simple example here may clarify the idea of Large Deviations. Consider a sequence of random variables representing nth average $\overline{S}_n$ of another sequence of random variables, i.e. $\overline{S}_n = (1/n)\sum_{j=1}^{n} y_j$, where $\{y_j\}$ are independent, identically distributed random variables. By the Law of Large Numbers, we know that $\overline{S}_n$ converges to the mean $\mathrm{E}(y_j) = \mu$ (assuming it is finite), i.e. $\mathrm{P}(|\overline{S}_n - \mu| > \varepsilon) \to 0$ as $n \to \infty$ for all $\varepsilon > 0$. It is often the case that this probability goes to zero exponentially fast, i.e. $\mathrm{P}(|\overline{S}_n - \mu| > \varepsilon) \approx K(\varepsilon, n)\exp[-nI(\varepsilon)]$, where K is slowly varying in n relative to the exponential term and I is a positive quantity (the form of K and I depends on the distribution of the y_j's). If this relation holds, we say that the sequence $\overline{S}_n$ obeys a Large Deviations principle with rate function I. Large Deviations theory concerns the identification of the function I in general, and K in some cases. The reason for calling this theory Large Deviations is that

[14] Welfare may improve even if such regulations do not provide new information. For example, there is a body of empirical evidence which suggests that many investors are functionally fixated (see Hand, 1990, for example.) Rather than evaluating all of the information contained in financial statements, such investors tend to fixate on the data inherent in the body of the financial statement reports such as the income statement and the balance sheet. In particular, they disregard financial statement footnotes not to speak of more esoteric sources of financial information such as proxy statements. Investors who sell their securities soley on the basis of the information contained in the body of financial statements are basically liquidity noise traders. Therefore, regulations that require or induce firms to disclose information in the body of the financial report rather than in the footnotes (or other data sources) may be welfare improving even if the information is already publicly available. FASB Statement No.123 is a case in point. *Inter alia,* this statement asks that firms account for employee stock-based compensation agreements in the income statement using option-pricing models to measure the fair value of the employee stock option. Yet, SEC proxy statements already include sufficient data to permit the investor to calculate employee stock option values on his/her own (see Murphy, 1996).

for a given positive ε and a large n, a large deviation from the value μ (or a rare event) has taken place if $|\overline{S}_n - \mu| > \varepsilon$.

A very general mathematical formulation is as follows. We consider a (complete, separable) metric space S and a family of probability measures P_ε on $\mathscr{B}(S)$, where $\varepsilon > 0$, and $\mathscr{B}(S)$ is the σ-algebra of Borel sets in S. We use the notation $\overline{A}$ and A° to denote the closure and the interior of the set A, respectively. Suppose that P_ε converges (weakly) to P_0 as $\varepsilon \to \infty$ (in the sense that for each set $A \in \mathscr{B}(S)$ such that $\mathrm{P}_0(\partial A) = 0, \quad \mathrm{P}_\varepsilon(A) \to \mathrm{P}_0(A)$, where $\partial A = \overline{A} \setminus A^\circ$ is the boundary of A; see Billingsley (1968) for the theory of weak convergence of probability measures). If $\mathrm{P}_0 = \delta_{x_0}$ for some $x_0 \in S$ (i.e. $\mathrm{P}_0(\{x_0\}) = 1$), then for each $A \in \mathscr{B}(S)$ such that $x_0 \notin \overline{A}, \quad \mathrm{P}_\varepsilon(A) \to 0$ (and for each $A \in \mathscr{B}(S)$ such that $x_0 \in A^\circ, \quad \mathrm{P}_\varepsilon(A) \to 1$). We say that P_ε satisfies a Large Deviations principle with rate function $I : S \to [0, \infty]$ if the rate of convergence is characterized by

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathrm{P}_\varepsilon(A) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathrm{P}_\varepsilon(A) \leq -\inf_{x \in \overline{A}} I(x) \qquad \text{(A.1)}$$

for all $A \in \mathscr{B}(S)$. For technical reasons, it is usually also required in the literature that the rate function I be lower semicontinuous and that the sets $\{x : I(x) \leq c\}$ be compact in S for each $c \in \mathbf{R}$.

Note that if $\inf_{x \in A^\circ} I(x) = \inf_{x \in \overline{A}} I(x) = \inf_{x \in A} I(x)$ then (A.1) implies

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathrm{P}_\varepsilon(A) = -\inf_{x \in A} I(x). \qquad \text{(A.2)}$$

In nontrivial cases (A.2) implies exponential convergence of $\mathrm{P}_\varepsilon(A)$ to zero (if $x_0 \notin \overline{A}$, i.e. if $0 < \inf_{x \in A} I(x) < \infty$), and in this sense the theory of Large Deviations provides rates in exponential convergence to zero of certain probabilities.

Note that the parameterization of $\{\mathrm{P}_\varepsilon, \varepsilon > 0\}$ with $\varepsilon \to 0$ can be replaced by $\{\mathrm{P}_n, n \geq 1\}$ where $n \to \infty$, or $\{\mathrm{P}_t, t \geq 0\}$ as $t \to \infty$. In practice, the probability measures P_ε (or P_n or P_t) are induced by families of random variables (random elements) with values in S. The simplest example, already mentioned above, is the case $S = \mathbf{R}$ with P_n being the distribution of $\overline{S}_n = \dfrac{1}{n} \sum_{j=1}^{n} y_j$, where $\{y_j, j \geq 1\}$ are i.i.d. random variables. Then, under the assumption of finiteness of the moment generating function $M(\theta) = \mathrm{E}(e^{\theta y_j}), \theta \in \mathbf{R}$, P_n satisfies a Large Deviations principle with rate function $I(x) = \sup_{\theta \in \mathbf{R}} [\theta x - \log M(\theta)]$ (Cramer's Theorem). This generalizes to $S = \mathbf{R}^k, k \geq 1$ when we take $\{y_j\}$ to be i.i.d. random vectors in $\mathbf{R}^k$ and consider $M(\theta) = \mathrm{E}(e^{\langle \theta, y_j \rangle}), I(x) = \sup_{\theta \in \mathbf{R}^k} (\langle \theta, x \rangle - \log M(\theta))$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^k$.

An important example of infinite dimensional case is $S = C[0, T]$ – the space of continuous functions on $[0, T]$ with the supremum norm. If we take B to be the standard Brownian motion on $[0, T]$ and P_ε to be the distribution of $\sqrt{\varepsilon} B(\cdot)$, then as $\varepsilon \to 0$, P_ε satisfies a Large Deviations principle with rate function

$$I(x) = \begin{cases} \frac{1}{2}\int_0^T (x'(t))^2 dt, & \text{if } x \in H[0,T], \\ \infty & \text{otherwise,} \end{cases}$$

where $H[0,T]$ is the space of functions which are absolutely continuous on $[0,T]$ with square integrable derivative (Schilder's Theorem).

The books of Varadhan (1984) and Deuschel and Stroock (1989) provide a mathematical treatment of general Large Deviations principles starting with the above examples, the theory of Level-2 Large Deviations used in Appendix B and the Wentzell-Freidlin theory applied in Appendix C.

Appendix B

B.1 Proof of Proposition 1

Define an occupation measure for the log of the price process $r_\varepsilon(t)$ as

$$\mu_t(A) = \frac{1}{t}\int_0^t \chi_A(r_\varepsilon(s))ds$$

where A is a measurable set of the real line **R**. Here $\mu_t(.)$ is viewed as an element in $M_1(\mathbf{R})$, the space of all probability measures on **R** endowed with the usual weak topology. The generator of the price process, which is a diffusion process, is given by

$$\mathscr{L} = \frac{1}{2}\varepsilon^2 \frac{\partial^2}{\partial r_\varepsilon^2} + \kappa(\theta - r_\varepsilon)\frac{\partial}{\partial r_\varepsilon}$$

$$= \frac{1}{2}\varepsilon^2 \exp\left\{\frac{\kappa(r_\varepsilon - \theta)^2}{\varepsilon^2}\right\}\frac{\partial}{\partial r_\varepsilon}\left(\exp\left\{-\frac{\kappa(r_\varepsilon - \theta)^2}{\varepsilon^2}\right\}\frac{\partial}{\partial r_\varepsilon}\right) . \tag{B.1}$$

This is a self adjoint differential operator with respect to the probability measure

$$m(dr_\varepsilon) = \frac{\sqrt{\kappa}}{\sqrt{\pi\varepsilon}}\exp\left\{-\frac{\kappa(r_\varepsilon - \theta)^2}{\varepsilon^2}\right\}dr_\varepsilon \tag{B.2}$$

and the Dirichlet form is given by

$$D(f) = \frac{\varepsilon^2}{2}\int\left(\frac{\partial f}{\partial r_\varepsilon}\right)^2 m(dr_\varepsilon) . \tag{B.3}$$

From the Donsker-Varadhan theory on level-2 large deviations (see Deuschel and Stroock(1989), Theorem 4.2.58 on p.133 and Section 5.3 on p.206 for details)

$$\lim\sup_{t\to\infty}\frac{1}{t}\log P\left(\{\mu_t \in C\}\right) \leq -\inf_{\nu\in C} I(\nu)$$

for any closed set $C \in M_1(\mathbf{R})$, and for any open set O

$$\lim\inf_{t\to\infty}\frac{1}{t}\log P(\{\mu_t \in O\}) \geq -\inf_{\nu\in C} I(\nu)$$

where

$$I(\nu) = \begin{cases} D(\sqrt{f}) \text{ if} & \nu(dr_\varepsilon) = fm(dr_\varepsilon) \\ \infty & otherwise \end{cases} .$$

We have that $\overline{r}_\varepsilon(t) = \int_{\mathbf{R}} r_\varepsilon \mu_t(dr_\varepsilon)$. To arrive at the large deviation estimates we need to apply the *Contraction Principle* (see Varadhan, 1984).[15] Note that even though the functional is not continuous, we can still invoke this principle subject to checking for exponential tightness.[16] Here it can be verified that this condition holds by following techniques similar to the ones outlined in Deuschel and Stroock (1989) (see exercise 2.1.20 on p. 47; also see Lemma 3.3.10 on p. 81 and exercise 3.3.12 on p. 84). Therefore, we can state that

$$\lim_{t\to\infty}\inf \frac{1}{t} \log P\left\{c < \overline{r}_\varepsilon(t) < d\right\} \geq -\inf_{c<x<d} I(x)$$

and

$$\lim_{t\to\infty}\sup \frac{1}{t} \log P\left\{c \leq \overline{r}_\varepsilon(t) \leq d\right\} \leq -\inf_{c<x<d} I(x)$$

for any real numbers $c < d$, and

$$I(x) = \inf_{\int f^2 m(dr_\varepsilon)=1, \int r_\varepsilon f^2 m(dr_\varepsilon)=x} D(f) . \tag{B.4}$$

The remaining part of the proof concerns the evaluation of the right hand side of (B.4) which is a conditional variational problem. To calculate this, consider the following Lagrangian functional

$$L(f) = D(f) + \lambda_1 \left(\int f^2 m(dr_\varepsilon) - 1\right) + \lambda_2 \left(\int r_\varepsilon f^2 m(dr_\varepsilon) - x\right)$$

subject to the constraints

$$\int f^2 m(dr_\varepsilon) = 1$$

$$\int r_\varepsilon f^2 m(dr_\varepsilon) = x .$$

The minimizing f_0 will satisfy the following three equations

$$[\mathscr{L}_I f_0] + \lambda_1 f_0 + \lambda_2 r_\varepsilon f_0 = 0$$

$$\int f_0^2 m(dr_\varepsilon) = 1$$

[15] The *Contraction Principle*, helps in establishing the *large deviations principle* and computing rate functions for transformations of random processes whose rate function is known.

[16] Exponential tightness permits results that satisfy what is known as *Strong Large Deviation Priniciple* (see Deuschel and Stroock, 1989, pp. 40-41). Informally, it makes the computation of the upper bound for the rate function more convenient, by allowing the focus to be only compact sets rather than checking every closed set. Intuitively, since large deviations deals with probabilistic behavior on the exponential scale, it is not surprising to encounter "exponential approximations" such as exponential tightness.

$$\int r_\varepsilon f_0^2 m(dr_\varepsilon) = x \ .$$

It is easy to see that

$$f_0 = \exp \frac{\kappa}{\varepsilon^2}\left[(x-\theta)r_\varepsilon - \frac{1}{2}(x^2-\theta^2)\right] \ .$$

It immediately follows form (B.4) that

$$I(x) = \inf_{\int f^2 m(dr_\varepsilon)=1, \int r_\varepsilon f^2 m(dr_\varepsilon)=x} D(f) = \frac{1}{2}\frac{\kappa^2}{\varepsilon^2}(x-\theta)^2 \ .$$

Since the rate function $I(x)$, for $\overline{r}(t)$ is continuous in x ,

$$\inf_{c<x<d} I(x) = \inf_{c\leq x\leq d} I(x) \ .$$

This then implies that the lower bound

$$\lim_{t\to\infty}\inf \frac{1}{t}\log P\{c < \overline{r}_\varepsilon(t) < d\}$$

and the upper bound

$$\lim_{t\to\infty}\sup \frac{1}{t}\log P\{c \leq \overline{r}_\varepsilon(t) \leq d\}$$

are identical. Hence

$$\lim_{t\to\infty} \frac{1}{t}\log P\{c \leq \overline{r}_\varepsilon(t) \leq d\} = \inf_{c\leq x\leq d} I(x) \ .$$

This proves the proposition.

B.2 Proof of Proposition 3

The proof of this proposition is similar to the proof of Proposition 1 above with equations $(B.5)$ to $(B.7)$ replacing equations$(B.1)$ to $(B.3)$.

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}\varepsilon^2 r_\varepsilon \frac{\partial^2}{\partial r_\varepsilon^2} + \kappa(\theta - r_\varepsilon)\frac{\partial}{\partial r_\varepsilon} \\ &= \frac{1}{2}\frac{1}{r_\varepsilon^\alpha e^{-\beta r_\varepsilon}}\frac{\partial}{\partial r_\varepsilon}\left(\{\varepsilon^2 r_\varepsilon (r_\varepsilon^\alpha e^{-\beta r_\varepsilon})\frac{\partial}{\partial r_\varepsilon}\right) \end{aligned} \tag{B.5}$$

where $\alpha = \frac{2\kappa\theta}{\varepsilon^2} - 1$ and $\beta = \frac{2\kappa}{\varepsilon^2}$. $\mathscr{L}$ is a self-adjoint operator with respect to the probability measure

$$m = \frac{\beta^{\alpha+1}}{\Gamma(\alpha)} r_\varepsilon^\alpha e^{-\beta r_\varepsilon} dr_\varepsilon \tag{B.6}$$

where $\Gamma(\alpha) = \int r_\varepsilon^\alpha e^{-r_\varepsilon} dr_\varepsilon$. The Dirichlet form is given by

$$D(f) = \frac{\varepsilon^2}{2} \int r_\varepsilon \left(\frac{\partial f}{\partial r} \right)^2 m(dr_\varepsilon) \ . \tag{B.7}$$

Applying the same Lagrangian procedure as before in Proposition 1, we get the minimum

$$f_0 = \exp \left[\frac{\kappa}{\varepsilon}(1 - \frac{\theta}{x})r_\varepsilon + m \right]$$

where m is a constant such that $f_0^2 m(dr)$ is a probability measure. Consequently, we have that

$$\begin{aligned} I(x) &= \inf_{\int f^2 m(dr_\varepsilon)=1, \int r_\varepsilon f^2 m(dr_\varepsilon)=x} D(f) \\ &= \frac{\kappa^2 x}{2\varepsilon^2} \left(\frac{\theta}{x} - 1 \right)^2 \ . \end{aligned}$$

The above function $I(x)$, is continuous and so the lower bound

$$\liminf_{t\to\infty} \frac{1}{t} \log P\{c < \overline{r}_\varepsilon(t) < d\} \geq - \inf_{c<x<d} I(x)$$

and the upper bound

$$\limsup_{t\to\infty} \frac{1}{t} \log P\{c \leq \overline{r}_\varepsilon(t) \leq d\} \leq - \inf_{c\leq x\leq d} I(x)$$

are identical. Therefore

$$\lim_{t\to\infty} \frac{1}{t} \log P\{c \leq \overline{r}_\varepsilon(t) \leq d\} = \inf_{c\leq x\leq d} I(x)$$

and the proposition is proved.

Appendix C

In the present section of the appendix we use Large Deviations principles for probability measures on $C[0, T]$ corresponding to diffusion processes determined by stochastic differential equations of the form

$$dX_\varepsilon(t) = \mu(X_\varepsilon(t))dt + \varepsilon\sigma(X_\varepsilon(t))dB(t). \tag{C.1}$$

This type of Large Deviations results are generally referred to as the Wentzell-Freidlin theory (cf. Freidlin and Wentzell, 1984).

We assume that $\mu : U \to \mathbf{R}$ and $\sigma : U \to \mathbf{R}$ are locally Lipschitz continuous functions on an open subset $U \subset \mathbf{R}$ (i.e. uniformly Lipschitz continuous on each compact subset of U). We further assume that for each $\phi \in H[0, T]$ and $x_0 \in U$ the equation

$$x'(t) = \mu(x(t)) + \sigma(x(t))\phi'(t)$$

$$x(0) = x_0$$

has a solution defined on $[0,T]$. (Local Lipschitz continuity guarantees uniqueness and local existence of solutions.) Assume also that the stochastic differential equation (C.1) has, for all sufficiently small ε and any initial condition $X_\varepsilon(0) = x_0 \in U$, a solution defined almost surely on $[0,\infty)$ with values in U. (Local existence and uniqueness of strong solutions is again a consequence of local Lipschitz continuity. An important special case when the assumption of global existence of a unique strong solution is satisfied is when $U = \mathbf{R}$ and $|\mu(x)| \vee |\sigma(x)| \leq M(1+|x|)$, for some constant $M < \infty$, cf. Ikeda and Watanabe, 1989, or Karatzas and Shreve, 1987).

Under the above assumptions, a Large Deviations principle holds for a family of stochastic processes defined by equation (C.1) with rate function $I_{0,T}$, where

$$I_{0,T}(x) = \inf\left\{\frac{1}{2}\int_0^T \left(\phi'(x)\right)^2 ds : \phi \in H[0,T] \text{ and}\right.$$
$$\left. x(t) = x_0 + \int_0^t \sigma(x(s))\phi'(s)ds + \int_0^t \mu(x(s))ds, 0 \leq t \leq T\right\}$$

and infimum over the empty set is ∞ (Azencott (1980), Priouret (1982)).

In the exit problem we apply the above Large Deviations principle to the set $A = \{x \in C[0,T] : \sup_{0\leq t\leq T} |x(t) - X_0(t)| > \delta\}$, where X_0 is the "equilibrium" trajectory corresponding to

$$\begin{aligned} dX_0(t) &= \mu(X_0(t))dt \\ X_0(0) &= x_0. \end{aligned}$$

We then obtain from (A.2)

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(X_\varepsilon \in A) = -\inf_{x\in A} I_{0,T}(x). \tag{C.2}$$

C.1 Proof of Proposition 2. Define the difference between r_ε and r_0 by p_ε. Then

$$\begin{aligned} dp_\varepsilon(t) &= -\kappa p_\varepsilon(t) + \varepsilon dB(t) \\ p_\varepsilon(0) &= 0, \end{aligned}$$

and the rate function for the Large Deviations problem associated with p_ε is

$$I_{0,t}(p) = \frac{1}{2}\int_0^t (p'(s) + \kappa p(s))^2 ds$$

(for functions p which are absolutely continuous on $[0,t]$). For $t > 0, x, y \in \mathbf{R}$ write

$$V(t,x,y) = \min_{p(0)=x,p(t)=y} I_{0,t}(p).$$

Then, by (C.2),

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^2 \log \mathrm{P}(\sup_{0\le t\le T}|r_\varepsilon(t)-r_0(t)|>\delta) &= \lim_{\varepsilon\to 0}\varepsilon^2 \log P(\sup_{0\le t\le T}|p_\varepsilon(t)|>\delta) \\ &= -\min_{\substack{0\le t\le T\\ |y|>\delta}} V(t,0,y).\end{aligned}$$

We first determine $V(t,x,y)$. This is a calculus of variations problem. Writing $F(t,p,p') = (p'(t)+\kappa p(t))^2$, the Euler's equation $F_p = \frac{d}{dt}F_{p'}$ becomes

$$p''(s) = \kappa^2 p(s),$$

$$p(0) = x, p(t) = y.$$

Solving, we obtain $p(s) = C_1 e^{\kappa s} + C_2 e^{-\kappa s}$, where

$$C_1 = \frac{y - xe^{-\kappa t}}{e^{\kappa t} - e^{-\kappa t}}, \qquad C_2 = \frac{xe^{\kappa t} - y}{e^{\kappa t} - e^{-\kappa t}}.$$

So $p'(s) + \kappa p(s) = 2\kappa C_1 e^{\kappa s}$. Now,

$$V(t,x,y) = \frac{1}{2}\int_0^t 4\kappa^2 C_1^2 e^{2\kappa s} ds = C_1^2 \kappa (e^{2\kappa t} - 1) = \kappa \frac{(e^{\kappa t}y - x)^2}{e^{2\kappa t} - 1}.$$

Hence,

$$V(t,0,y) = \kappa y^2 \frac{e^{2\kappa t}}{e^{2\kappa t} - 1}.$$

It is now easy to see that

$$\min_{\substack{0\le t\le T\\ |y|>\delta}} V(t,0,y) = \kappa\delta^2 \frac{e^{2\kappa T}}{e^{2\kappa T} - 1}.$$

C.2 Proof of Proposition 4

The rate function associated with the family of diffusion processes defined by (5) is

$$I_{0,t}(r) = \inf\Big\{ \frac{1}{2}\int_0^t \left(\phi'(s)\right)^2 ds : \phi \in H[0,t] \text{ and}$$
$$r(s) = \theta + \int_0^s \sqrt{r(u)}\,\phi'(u)du + \int_0^s \kappa(\theta - r(u))du, 0 \le s \le t \Big\},$$

which for absolutely continuous, nonnegative r is

$$I_{0,t}(r) = \frac{1}{2}\int_0^t \frac{(r' - \kappa(\theta - r))^2}{r} ds. \tag{C.3}$$

(Note that the integrand is well defined almost everywhere on $[0, t]$.)

Proceeding as in the proof of Proposition 2, we seek $\min_{\substack{0 \le t \le T \\ |y-\theta|>\delta}} V(t, \theta, y)$, where

$$V(t, x, y) = \min_{r(0)=x, r(t)=y} I_{0,t}(r).$$

After some tedious calculations involving only calculus and algebra, the rate function is derived to be

$$\min_{\substack{0 \le t \le T \\ |y-\theta|>\delta}} V(t, \theta, y) = (C + 2\kappa\sqrt{\theta C})\frac{e^{\kappa t} - 1}{2\kappa} - 2\kappa\theta \log \frac{\sqrt{C}(e^{\kappa t} - 1) + 2\kappa\sqrt{\theta}}{2\kappa\sqrt{\theta}}, \quad \text{(C.4)}$$

where

$$C = \frac{2\kappa^2[\theta(e^{\kappa t} + 1)^2 + 2e^{\kappa t}\delta - (e^{\kappa t} + 1)\sqrt{(e^{\kappa t} + 1)^2\theta^2 + 4e^{\kappa t}\theta\delta}]}{(e^{\kappa t} - 1)^2}.$$

Details of the calculations are available on request.

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