We study in the classic newsvendor framework inventory competition with spillover demand between risk-averse duopoly firms. We establish existence and uniqueness of a Nash equilibrium and characterize the sensitivity of the equilibrium order quantities to an increase in both firms’ risk-aversion. We report two sets of results. First, we provide fairly general necessary and sufficient conditions under which one of the firms increases its equilibrium quantity in response to higher risk-aversion. These conditions reflect two countervailing effects of increasing risk-aversion, a direct own risk-aversion effect, and an indirect demand spillover effect. Second, we report more specific results for the case where both firms have identically distributed binary primary demand and constant absolute risk-aversion (CARA) utility. We identify which of the firms responds with a higher equilibrium order to an increase in risk-aversion, depending on the initial equilibrium and on the competitors’ under- to overstocking cost ratios, initial risk-aversion, and demand spillover fractions. We further show how these effects depend on primary demand correlation.

1 Introduction

It is understood and experimentally verified that inventory purchase decisions may not be consistent with expected profit maximization. For example, experimental evidence suggests that inventory planners for high profit products trade off lower expected profits in return for downside protection against possible losses (Schweitzer and Cachon 2000). The level and nature of risk attitudes may further vary depending on various factors such as beliefs about market volatility, the availability of cash/credit, and the cost of financing. While the importance of risk aversion for inventory decisions is recognized, to our knowledge there are no studies that investigate the interaction of risk aversion and competition on inventory decisions. This paper contributes to filling this gap.

We study the newsvendor game among risk-averse competitors who each place a single order for one substitutable product. We first study the benchmark case of two single-product firms with general risk preferences. We assume that the product costs and prices are fixed. The only source of uncertainty is demand. The firms order simultaneously prior to observing demand, aiming to maximize their expected utility. Each firm faces its primary demand, plus spillover demand if its competitor stocks out (a deterministic fraction of her unmet demand). We characterize the Nash
equilibrium orders as functions of the problem parameters and investigate their sensitivity to risk aversion levels. We show that one of the firms may raise its order if both firms become more risk averse. This suggests that ignoring the presence of competition may lead to wrong recommendations on how to adjust to increasing risk aversion.

We show that higher risk aversion has two opposite effects on equilibrium orders:

1. The direct own risk aversion effect, which causes a firm to reduce its order quantity.
2. The indirect demand substitution or demand spillover effect, which causes the firm to increase its order quantity.

To put these results in context, it is well known that the optimal order quantity of a single-product monopoly decreases in risk aversion; e.g., see Eeckhoudt et al. (1995). In this case, there is no demand substitution effect among products. Choi et al. (2008, 2011) study a multi-product risk averse monopoly newsvendor without focusing on demand substitution effects. The interaction between the own risk aversion effect and the demand substitution effect in the presence of competition appears not to have been studied so far.

Next, to gain specific insights on how risk aversion affects the equilibrium depending on the model parameters, we assume that firms’ risk preferences exhibit constant absolute risk aversion (CARA Utility Function) and each firm’s primary demand is low or high. Initially, we consider the case uncorrelated primary demands. Later we relax this assumption.

We identify which firm raises its order quantity as both firms become more risk averse, when the firms differ in one of three attributes: the profitability of their product, measured by their under-to overstocking cost ratio; their risk aversion level; and their spillover demand fraction.

**Different profitability ratio.** We show for uncorrelated demand that either the more profitable or the less profitable firm increases its order quantity as both become more risk averse.

**Different initial risk aversion.** We show that either the initially more risk averse or the less risk averse firm may increase its order quantity as both firms become more risk averse. Which case applies also depends on the profitability of the firms. In particular, if the profitability ratio is low, only the initially less risk averse firm increases its order quantity as both firms become more risk averse. If profitability is high, the less risk averse firm increases its order quantity if its competitor’s initial risk aversion is low or quite significant, and the more risk averse firm increases its order quantity if its competitor’s initial risk aversion is in the intermediate range.

**Different spillover fractions.** We show that when both firms are identical except for their demand spillover rate, either the firm with the higher or the one with the lower spillover rate may increase its order quantity as both firms become more risk averse. Which case applies also depends on the profitability of the firms. When the profitability is very low the firm with less spillover
demand will increase its order quantity as both firms become more risk averse. However, when the profitability is in the low to intermediate range, the firm with more spillover demand increases its order quantity as both firms become more risk averse. When both firm are highly profitable neither of the firms increase their order quantity as both become more risk averse.

Correlated demand. We also study the effect of demand correlation. We show that when the demand is perfectly positively correlated, only the more profitable firm or the less risk averse firm may increase its order quantity as both firms become more risk averse. The only case in which a firm increases its order quantity as both firms become more risk averse, is when it is willing to order the high demand plus the spillover from its competitor. To order such a significant order quantity, the firm should either be significantly more profitable or initially less risk averse. When demand is perfectly negatively correlated, only the less profitable or the more risk averse firm may increase its order quantity. In this case the firm with lower order quantity at the equilibrium increases its order quantity as both firms become more risk averse. To have a lower order quantity at the equilibrium, the firm must have lower profitability or higher initial risk aversion.

We also show that under perfect positive or perfect negative correlation, if firms only differ in their spillover fractions, neither firm increases its order quantity in response to higher risk aversion.

Literature and positioning. All the early and much of the recent inventory control literature assumes risk neutral decision makers. See Khouja (1999). However, several papers study the impact of risk aversion on inventories for a monopoly. Berman and Schnabel (1986) were among the first to study the impact of risk aversion on the order quantity under a mean-variance utility function. Eeckhoudt et al. (1995) study a monopoly newsvendor under various utility functions. Agrawal and Seshadri (2000) study the impact of uncertainty and risk aversion on price and order quantity. Van Mieghem (2007) studies newsvendor networks with many products and resources under mean variance utility function. The paper discuss how operational strategies (operational hedging) may reduce total risk and hence create value. It shows that compared to a risk neutral newsvendor, a risk averse newsvendor may invest more on resources to reduce the profit variance and mitigate the risk in the network. Chen et al (2007) study the effect of risk aversion in multi-period inventory models. Choi et al (2011) study a multi-product risk averse monopoly newsvendor under general law-invariant coherent measures of risk. They show that for identical products with independent demands, increased risk aversion lead to decreased order quantities. Furthermore, they show numerically that the same result holds for a two-product newsvendor with positive correlation. However, for a two-product newsvendor with negatively correlated demand, they show (also numerically) that increased risk aversion may result in a greater order quantity for one of
the products, compared to a risk neutral newsvendor. Choi and Ruszczyński (2011) also study the multi-product risk averse monopoly newsvendor with exponential utility function.

There are also several studies of inventory competition assuming risk-neutral decision-makers. While competition clearly affects risk, existing studies consider risk aversion and competition only in isolation, leaving open many questions on their interaction. Parlar (1988) was first to study a duopoly in the context of the classic newsvendor framework. He showed the existence and uniqueness of Nash equilibrium for the risk neutral duopoly newsvendor game. Lippman and McCardle (1997) study the effect of the initial allocation rule on the equilibrium. Netessine and Rudi (2003) study centralized and competitive models for multiple products with demand substitution.

Another set of related papers consider both pricing and inventory decisions. Ayden and Porteus (2008) seek optimal inventory levels and prices under price based substitutions but not stock-out based substitutions. Zhao and Atkins (2008) study a more general case. They obtain optimal inventory levels and price for N vendors selling substitutable products. However, they consider both price based and stock-out based substitutions. Hu et al. (2011) extended Zhao and Atkins (2008). They consider two-period game in contrast to Zhao and Atkins (2008) single period problem and also endogenized the consumer’s switching behavior upon a stockout. We refer to Ayden and Porteus (2008), Zhao and Atkins (2008), and Hu et al. (2011) for further references.

The interaction between risk aversion and competition also caught the interest of researchers in economics and finance. However, those papers ignore inventory decisions and supply constraints; see, e.g., Eldor and Zilcha (1990) and Asplund (2002) for studies of risk averse oligopoly in the absence of inventory constraints.

2 Basic Model

We consider a single-period newsvendor game in stocking decisions between risk-averse duopoly firms which sell substitutable products at fixed prices. Firms are indexed by \( i \in \{A, B\} \), and quantities pertaining to firm \( i \) are denoted by a superscript. Each firm sells a single product. Firms make stocking decisions independently and simultaneously. Firm \( i \) orders \( O^i \) units by the beginning of the sales period at unit cost \( c^i \), sells its product at unit price \( p^i \) and disposes of leftover inventory at unit salvage value \( s^i \), where \( p^i > c^i > s^i > 0 \). Define the underage cost \( C^i_u = p^i - c^i \), the overage cost \( C^i_o = c^i - s^i \), and let \( K^i = C^i_u + C^i_o \) denote their sum.

Each firm’s total demand equals the sum of her own primary demand plus spillover demand from her competitor. Primary demands do not depend on the order quantities while each firm’s spillover demand equals a deterministic fraction of her competitor’s unsatisfied primary demand. Firm \( i \)’s
spillover demand equals a fraction $b^i > 0$ of her competitor’s unsatisfied primary demand, where we allow $b^i \neq b^j$. This model of competition based on consumer-driven substitution is frequently used in the literature (McGillivray and Silver 1978, Parlar and Goyal 1984, Noonan 1995). It describes situations in which consumers first try to buy the product they prefer, e.g., based on brand loyalty, perceived quality or shopping convenience, and then may substitute a similar product if their first choice is stocked out.

The random variable $D^j(O^j)$ denotes firm $i$’s total demand as a function of $O^j$ for $j \neq i$. Let $d_{xy}^i(O^j)$ denote a total demand realization. We assume that the joint probability distribution of Firm A and B’s demand is given, i.e. Firm A and B experience primary demand $d_x^A$ for $x = 1, 2, ..., N^A$ and $d_y^B$ for $y = 1, 2, ..., N^B$ respectively where $d_1^A < d_2^A < ... < d_{N^A}^A$ and $d_1^B < d_2^B < ... < d_{N^B}^B$. Therefore firm A faces a total demand realization $d_{xy}^i(O^j) = d_x^A + b^A (d_y^B - O^B)^+$ with probability $q_{xy}$. Note that a given total demand realization may correspond to multiple pairs of primary demand realizations. In this case at most one of these pairs involve no spillover demand, and the other pair(s) involve spillover demand. All of these pairs with spillover continue to yield the same total demand in response to small changes in the rival’s order. Notice that the results of Sections 3 and 4 generalize in a natural way to the case of arbitrary continuous primary demand distributions. However, we present them for discrete distributions to build on them for the special case of binary demand in Sections 5 and 6.

Remark. This definition of total demand can accommodate a number of initial demand allocation rules, including “Independent Random Demands”, “Deterministic Splitting”, “Simple Random Splitting” and “Incremental Random Splitting”, as defined in Lippman and McCardle (1997).

Let $\Pi^i(O^j, d)$ denote firm $i$’s payoff as a function of her order quantity $O^j$ and the demand realization $d$. It satisfies

$$\Pi^i(O^j, d) = \left((C_u^i + C_o^i)d - C_o^j O^j\right) 1\{d < O^j\} + C_o^i O^j 1\{d \geq O^j\}. \quad (1)$$

As noted above we consider risk-averse firms.

Assumption 1. We assume the utility function, $u^i(x)$, has a general form such that $u''(x) > 0$, $u'''(x) < 0$, and $u'''(x)/u''(x)$ is a non-decreasing function of $x$.

Assumption 1 corresponds to non-increasing Arrow-Pratt absolute risk aversion coefficient. Note that Assumption 1 with regard to the utility function is not restrictive and holds for many classes of utility functions, such as Constant Absolute Risk Aversion (CARA) utility function

$$U(x) = 1 - \exp(-Rx),$$
Constant Relative Risk Aversion (CRRA) utility function

\[ U(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } R > 0, R \neq 1 \\ \ln x & \text{if } R = 1 \end{cases} \]

and Hyperbolic Absolute Risk Aversion (HARA) utility function

\[ U(x) = \frac{1 - \gamma}{\gamma} \left( \frac{ax}{1 - \gamma} + b \right)^\gamma, \quad a > 0, \gamma < 1, \text{ and } \frac{ax}{1 - \gamma} + b > 0. \]

Remark. (1) As \( \gamma \to -\infty \) and \( b = 1 \), HARA utility function converge to CARA. (2) When \( b = 0 \), and \( \gamma \leq 1 \), the HARA utility function becomes CRRA. When \( b = 0 \) and \( \gamma < 1 \), HARA utility function is equivalent to the following form of CRRA utility function

\[ U(x) = \frac{x^{1-R}}{1-R} \]

and when \( b = 0 \) and \( \gamma \to 1 \), HARA utility function is equivalent to the logarithmic form of CRRA utility function.

Note that as pointed out before, in case of HARA utility function, we must have:

\[ \frac{ax}{1 - \gamma} + b > 0 \iff x > -\frac{b(1 - \gamma)}{a}. \]

Since the payoff is:

\[ x = ((C_u + C_o)d - C_o)1\{d < O\} + C_u O 1\{d \geq O\} \]

we must have:

\[ (C_u + C_o)d_{\text{min}} - C_o O > \frac{b(1 - \gamma)}{a} \iff \left( \frac{C_u}{C_o} + 1 \right)d_{\text{min}} + \frac{b(1 - \gamma)}{aC_o} > O \]

and

\[ O > -\frac{b(1 - \gamma)}{aC_u}. \]

By choosing appropriate parameter for HARA utility function the following range is not a restrictive for order quantity:

\[ \left( \frac{C_u}{C_o} + 1 \right)d_{\text{min}} + \frac{b(1 - \gamma)}{aC_o} > O > -\frac{b(1 - \gamma)}{aC_u}. \]

Firm \( i \)'s expected utility function given the order of firm \( j \neq i \) is defined as

\[ U^i(O^i|O^j) \equiv E \left[ u^i \left( \Pi^i \left( O^i, D^j(O^j) \right) \right) \right] = \sum_x \sum_y q_{xy} u^i \left( \Pi^i \left( O^i, d^j_{xy}(O^j) \right) \right). \]  

(2)

In detail this gives

\[ U^i(O^i|O^j) = \sum_x \sum_y q_{xy} \left[ u^i \left( K^i d^j_{xy}(O^j) - C_o^i O^j \right) 1\{d^j_{xy}(O^j) < O^i\} + u^i \left( C_u^i O^i \right) 1\{d^j_{xy}(O^j) \geq O^i\} \right]. \]
Each firm chooses the order quantity that maximizes the expected utility of its payoff. Let 
\( f^A(O^B) \triangleq \arg \max_{O^A} U^A(O^A|O^B) \) denote firm \( A \)'s best response as a function of her competitor’s order quantity \( O^B \). We denote a Nash equilibrium by \( O^* = (O^{A*}, O^{B*}) \). We denote a Nash equilibrium by \( Q^* = (Q^{A*}, Q^{B*}) \). The Nash equilibrium for a two-player, continuous game is a pair \((Q^{A*}, Q^{B*})\) with the property that:

\[
U^A(Q^{A*}|Q^{B*}) \geq U^A(Q^A|Q^{B*}) \text{ for all } Q^A,
\]

\[
U^B(Q^{A*}|Q^{B*}) \geq U^B(Q^{A*}|Q^{B}) \text{ for all } Q^B.
\]

### 3 Equilibrium Characterization

In this section we initially obtain the unique best response function for a given competitor’s order quantity, and then we show the existence and uniqueness of the equilibrium.

**Best Response.** The best response problem of firm \( i \) is mathematically equivalent to the problem of a monopoly facing an exogenous random demand \( D^j(O^j) \). For ease of exposition we first characterize the solution of this problem for general utility function with \( u' > 0, u'' < 0 \) and \( u'''(x)/u''(x) \) is a non-decreasing function of \( x \). We then map it to the duopoly best response. Dropping superscripts, let

\[
U(O) \triangleq E[u(\Pi(O,D))] = \sum_{i=1}^N q_i \cdot u(\Pi(O,d_i))
\]

(3)
denote the expected utility of a monopoly firm facing exogenous random demand \( D \) where \( d_i > 0 \) and \( q_i \triangleq P(D = d_i) \). Lemma 1 summarizes basic properties of \( U(O) \) and \( O^* \triangleq \arg \max_{O \geq 0} U(O) \).

**Lemma 1** The expected utility function \( U \) and its maximizer \( O^* \) have the following properties.

1. The expected utility satisfies \( U(O) = U_k(O) \) for \( O \in [d_k, d_{k+1}] \) where \( d_0 \triangleq 0, d_{N+1} \triangleq \infty \) and

\[
U_k(O) \triangleq \sum_{i=1}^k q_i \cdot u((C_u + C_o)d_i - C_o O) + \left(1 - \sum_{i=1}^k q_i\right) \cdot u(C_u O), \quad k = 0, 1, 2, ..., N.
\]

(4)

It is continuous and strictly concave. For \( k \in \{1, 2, ..., N\}, U \) has left and right derivatives \( U'_- \) and \( U'_+ \) at \( d_k \), is twice continuously differentiable on \((d_k, d_{k+1})\) and satisfies \( U''(O) = U''_k(O) < 0 \) if \( O \in (d_k, d_{k+1}) \) and \( U'_{-}(d_k) = U'_{-}(d_{k-1}) > U'_{+}(d_k) = U'_{+}(d_k) \).

2. The optimal order quantity \( O^* \) is unique. It satisfies \( O^* \in [d_1, d_N] \) and

\[
O^* = \max \left\{ O \geq 0 : \sum_{i=1}^N 1\{d_i < O\} \cdot q_i \cdot \left(\frac{u'((C_u + C_o)d_i - C_o O)}{u'(C_u O)} - C_o + C_u\right) \leq C_u \right\}.
\]

(5)
Lemma 1 clearly applies for any demand distribution. For given \( O^j \), the total demand distribution \( D^i(O^j) \) is induced and hence the best response function of firm \( i \), \( f^i(O^j) \), is well defined and we can translate the order prescription of Lemma 1 into the duopoly best response.

**Lemma 2** The expected utility function \( U^i(O^i|O^j) \) and the best response function \( f^i(O^j) \) of firm \( i \) are continuous and piecewise differentiable in \( O^j \).

1. If firm \( j \)'s order quantity is less than its own maximum primary demand i.e. \( O^j < d^j_{N_j} \), and firm \( i \)'s order quantity is larger than or equal to it minimum total demand that involves spillover demand i.e.

\[
O^i \geq \min_x \left( \frac{\hat{d}_x^i}{b} \left( \hat{d}^i \left( d^i_x, O^j \right) - O^j \right) \right) \quad \text{where} \quad \hat{d}^i \left( d^i_x, O^j \right) = \min_z \left[ d^i_z : 1_{\{d^i_z > O^j, q_{x,z} > 0\}} \right],
\]

then \( U^i(O^i|O^j) \) is strictly decreasing in \( O^j \). Otherwise, \( U^i(O^i|O^j) \) is constant in \( O^j \).

2. The left derivative \( f^i_{-} \) and the right derivative \( f^i_{+} \) of the best response function satisfy

\[
0 \geq f^i_{-}, f^i_{+} \geq -b^i. \tag{6}
\]

Note that if the primary demands of both firms were uncorrelated, \( \hat{d}^i \left( d^i_x, O^j \right) = \min_z \left[ d^i_z : d^i_z > O^j \right] \).

**Equilibrium Existence and Uniqueness.**

**Proposition 1** There exists a Nash equilibrium. If \( b^A b^B < 1 \) then it is unique.

Remark and assumption. If \( b^A = b^B = 1 \), the equilibrium is not necessarily unique. There may exist a continuum of equilibria whereby both firms order minimum primary demand plus spillover. For simplicity we henceforth assume that \( b^A b^B < 1 \).

### 4 Impact of Risk Aversion on Equilibrium Orders

Having established existence and uniqueness of the equilibrium, we now turn to the main question of this paper: how does a change in risk aversion affect the equilibrium order quantities? We address this question by studying the comparative statics of the order equilibrium with respect to an infinitesimal increase in both firms’ risk aversion parameters. Note that based on Pratt’s Theorem (see Pratt (1964)), increasing risk aversion is a concave transformation of the utility function which is equivalent to an increasing Arrow-Pratt absolute risk aversion coefficient for given payoff realization. For utility functions with one parameter such as CARA or CRRA to increase Arrow-Pratt absolute risk aversion coefficient, there is only one parameter that can be
changed. However for utility function with multiple parameter such as HARA, increasing Arrow-Pratt absolute risk aversion coefficient can be potentially achieved by changing many parameters. In such cases, we assume that all parameters are fixed except for one.

Let \( \mathbf{R} = (R^A, R^B) \) denote the vector of initial risk aversion parameters and \( Q^*(\mathbf{R}) = (Q^{A*}(\mathbf{R}), Q^{B*}(\mathbf{R})) \) be the corresponding equilibrium. Note that \( Q^*(\mathbf{R}) \) is only piecewise differentiable, but it is continuous, and its left and right partial derivatives with respect to \( \mathbf{R} \) are well defined.

Our analysis focuses on the question: which firm, if any, increases its equilibrium order quantity as both firms’ risk aversion increases, and under what conditions? Mathematically, under what conditions does the following hold?

\[
\frac{\partial Q^{i*}_+(\mathbf{R})}{\partial R^A} + \frac{\partial Q^{i*}_+(\mathbf{R})}{\partial R^B} > 0, \text{ for } i = A \text{ or } i = B, \tag{7}
\]

where \( \partial Q^{i*}_+(\mathbf{R})/\partial R^A, \partial Q^{i*}_+(\mathbf{R})/\partial R^B \) denote the right partial derivatives of \( Q^{i*}(\mathbf{R}) \).

Theorem 1 and Corollary 1 establish necessary and sufficient conditions for (7) in general form. This analysis formally identifies two countervailing effects that determine the sensitivity of each firm’s equilibrium order to an increase in risk aversion, the own risk aversion and the demand substitution effect.

**Theorem 1** Fix a risk aversion vector \( \mathbf{R} > 0 \) and the corresponding equilibrium \( Q^*(\mathbf{R}) \). Let the function \( g^i(Q^{-i}, R^i) \) be a piece of the firm \( i \) best response function \( f^i \) that satisfies the equilibrium conditions

\[
Q^{i*}(\mathbf{R}) = g^i(Q^{-i*}(\mathbf{R}), R^i) = f^i(Q^{-i*}(\mathbf{R}), R^i), \ i \in \{A, B\}. \tag{8}
\]

The firm \( i \) equilibrium order increases as both risk aversion parameters increase, i.e.,

\[
\frac{\partial Q^{i*}_+(\mathbf{R})}{\partial R^A} + \frac{\partial Q^{i*}_+(\mathbf{R})}{\partial R^B} > 0, \tag{9}
\]

if and only if the following two conditions hold:

1. The functions \( g^i(Q^{-i}, R^i) \) and \( g^{-i}(Q^i, R^{-i}) \) agree with the firms’ best response functions for larger risk aversion in some neighborhood of \( \mathbf{R} \): for some \( \delta > 0 \) and \( \delta \in [0, \delta] \),

\[
Q^{i*}(\mathbf{R}(\delta)) = g^i(Q^{-i*}(\mathbf{R}(\delta)), R^i(\delta)) = f^i(Q^{-i*}(\mathbf{R}(\delta)), R^i(\delta)), \ i \in \{A, B\}, \tag{10}
\]

where \( \mathbf{R}(\delta) = \mathbf{R} + \delta \mathbf{e} \) and \( \mathbf{e} = (1, 1) \).

2. The partial derivatives of \( g^i(Q^{-i}, R^i) \) and \( g^{-i}(Q^i, R^{-i}) \) satisfy

\[
\frac{\partial g^{-i}(Q^{i*}(\mathbf{R}), R^{-i})}{\partial R^{-i}} \frac{\partial g^i(Q^{-i*}(\mathbf{R}), R^i)}{\partial Q^{-i}} > - \frac{\partial g^i(Q^{-i*}(\mathbf{R}), R^i)}{\partial R^i}. \tag{11}
\]
Condition (10) is technical in nature: it is only required since the best response functions are piecewise functions. The important condition is (11). It formally identifies two countervailing effects that determine the sensitivity of each firm’s equilibrium order to an increase in both firms' risk aversion level:

1. The direct own risk aversion effect. An increase in firm $i$’s own risk aversion reduces its order, holding its rival’s order and risk aversion level fixed. The RHS of (11) measures the magnitude of this order reduction.

2. The indirect demand substitution effect. An increase in the risk aversion level of firm $i$’s rival increases its own order, holding its own risk aversion level fixed. The LHS of (11) measures the magnitude of this effect. It is the product of two factors, the reduction of the rival’s order size in response to its higher risk aversion, and the increase in firm $i$’s order to capture the resulting higher spillover demand.

Firm $i$’s equilibrium order increases in response to higher risk aversion if and only if the demand substitution effect is larger than the own risk aversion effect.

The following Corollary provides two general observations that follow from Theorem 1.

**Corollary 1** Fix a risk aversion vector $R > 0$ and the corresponding equilibrium $Q^*(R)$.

1. If firms are symmetric, then their equilibrium orders do not increase as both risk aversion parameters increase, i.e.,

$$\frac{\partial Q^i_+ (R)}{\partial R^A} + \frac{\partial Q^{-i}_+ (R)}{\partial R^B} \leq 0, \ i \in \{A, B\}.$$  \hspace{1cm} (12)

2. If the firm $i$ equilibrium order increases as both risk aversion parameters increase, i.e.,

$$\frac{\partial Q^i_+ (R)}{\partial R^A} + \frac{\partial Q^{-i}_+ (R)}{\partial R^B} > 0,$$

then the equilibrium order of its rival does not increase.

These results are quite intuitive. Since each firm only receives a fraction of its rival’s unmet demand, it finds it optimal to increase its order by less than the reduction in its rival’s order; see Part 2 of Lemma 2. Therefore, (11) can only hold for a firm if its own risk aversion effect is strictly smaller in magnitude than that of its rival. Clearly, this can only hold for one of the firms. It holds for neither firm if they are symmetric, for then the initial equilibrium is symmetric and the own risk aversion effect is equal for both firms.

To put Theorem 1 and Corollary 1 in context, it is well known that the optimal order quantity of a single-product monopoly decreases in the risk aversion level; e.g., see Eeckhoudt et al. (1995).
In this case, there is no demand substitution effect. Choi et al. (2008, 2010) study a multi-product risk averse monopoly newsvendor without focusing on demand substitution effects. The interaction between the own risk aversion effect and the demand substitution effect in the presence of competition appears not to have been studied so far.

5 Uncorrelated Binary Primary Demand with CARA Utility

To obtain and understand specific insights with regard to the impact of risk aversion, in this section we study the problem for a specific demand distribution and utility function. We assume that the primary demand distributions are binary, uncorrelated and identically distributed and each firm has a Constant Absolute Risk aversion (CARA) utility function. The CARA utility function is commonly used in the literature. It offers tractability, which helps us obtain specific insights into the impact of risk aversion on equilibrium orders. In Section 5.1, first, in light of Lemma 1, we obtain the best response function, and then based on that, we identify 17 distinct cases as possible equilibria, each corresponding to a particular subset of the order space. We, then, in Section 5.2, study the impact of risk aversion. In particular, we identify 6 cases out of those 17 cases that one of the firm may increase its order quantity as both firms become more risk averse. In Sections 5.3 to 5.5, we specify which firm raises its order quantity as both firms become more risk averse for three scenarios of asymmetric firms that differ in one of the following attributes: the profitability of their product, measured by their under- to overstocking cost ratio, their risk aversion level, and their spillover demand fraction. Also note that latter in Section 6 we relax the assumption of uncorrelated primary demand and study the impact of primary demand correlation.

5.1 Model and Equilibrium Characterization

In this section, we assume that the primary demand of each firm is \( d_L \) with probability \( q_L \) and \( d_H > d_L \) with probability \( q_H = 1 - q_L \). The assumption of binary primary demand yields explicit equilibrium characterizations, which allows us to gain specific insights on how risk aversion affects the equilibrium depending on the model parameters.

Let \( d_{xy}^i(Q^j) \) denote a demand realization and \( q_{xy} = q_x q_y \) the corresponding probability where \( x, y \in \{L, H\} \), \( x \) is firm \( i \)'s primary demand realization and \( y \) her competitor's. We assume w.l.o.g. that \( Q^A, Q^B \geq d_L \): firms have no incentive to order less than \( d_L \). Since the primary demands are
uncorrelated, the p.d.f. of firm $A$ total demand $D^A(Q^B)$ satisfies:

$$P(D^A(Q^B) = d) = \begin{cases} 
q_{LL} = q^2_L, & d = d^A_{LL}(Q^B) = d_L \\
q_{LH} = q_Lq_H = q_L(1 - q_L), & d = d^A_{LH}(Q^B) = d_L + b^A[d_H - Q^B]^+ \\
q_{HL} = q_Hq_L = (1 - q_L)q_L, & d = d^A_{HL}(Q^B) = d_H \\
q_{HH} = q_H^2 = (1 - q_L)^2, & d = d^A_{HH}(Q^B) = d_H + b^A[d_H - Q^B]^+ 
\end{cases} \quad (14)$$

If $Q^B < d_H$, firm $A$ receives spillover demand if her competitor’s primary demand is high. Otherwise, she adopts a monopoly strategy.

As mentioned before, we model the preferences of each firm by a constant absolute risk aversion (CARA) utility function, i.e. $u^i(x) = 1 - \exp(-R^i x)$ where $R^i > 0$ is the risk aversion rate.

If $Q^j \geq d_H$ or $b^i = 0$ then firm $i$ receives no spillover demand, $d^i_{xL}(Q^j) = d^i_{xH}(Q^j) = d_x$ and $q_{xL} + q_{xH} = q_x$ for $x \in \{L, H\}$. In this case

$$U^i(Q^j | Q^i) = \sum_{x \in \{L, H\}} q_x \left[ u^i(K^i d_x - C^i_u Q^i) 1\{d_x < Q^i\} + u^i(C^i_u Q^i) 1\{d_x \geq Q^i\} \right].$$

First consider how Lemma 1 specializes for CARA utility given by $u(x) = 1 - \exp(-R x)$.

**Lemma 3** To compute $Q^*$ in Lemma 1, consider equation (4) and let $Q^*_k \triangleq \arg \max_{Q^k} U^i_k(Q)$ for $k \in \{1, 2, ..., N - 1\}$ where $Q^*_k$ is the unique solution of $U^i_k(Q) = 0$ and $Q^*_0 \triangleq \infty > Q^*_1 > Q^*_2 > ... > Q^*_{N-1}$. The optimal order quantity is $Q^* = \min(Q^*_m, d_{m+1})$ where $m \triangleq \max\{0 \leq k \leq N - 1 : Q^*_k \geq d_k\}$.

Letting $K = C_u + C_o$, the maximizers $Q^*_k$ from above Lemma

$$U^i_k(Q^*_k) = 0 \Leftrightarrow Q^*_k = \frac{1}{RK} \ln \left( \frac{C_u}{C_o} \frac{1 - \sum_{i=1}^k q_i}{\sum_{i=1}^k q_i \exp(-RK d_i)} \right), \quad k \in \{0, 1, 2, ..., N - 1\}, \quad (15)$$

where $Q_0 = \infty$. The optimal order quantity $Q^*$ equals $Q^*_k$ if and only if $d_k \leq Q^*_k \leq d_{k+1}$, or equivalently:

$$\sum_{i=1}^k q_i \exp(-RK d_i) \leq \frac{C_u}{C_o} \leq \frac{1 - \sum_{i=1}^k q_i}{\sum_{i=1}^k q_i \exp(-RK d_i)} \exp(RK d_{k+1}). \quad (16)$$

The optimal order is $Q^* = \min(Q^*_m, d_{m+1})$ where

$$m = \max\left\{ 0 \leq k \leq N - 1 : \frac{1 - \sum_{i=1}^k q_i \exp(-RK d_i)}{\sum_{i=1}^k q_i} \exp(RK d_k) \leq \frac{C_u}{C_o} \right\}. \quad (17)$$

The optimal order is fully specified by the model primitives: underage and overage costs, the demand distribution and the risk aversion parameter. To complete the translation to the best response, add superscripts and replace the generic demand distribution by (14), to obtain the following Lemma. For simplicity it suppresses the dependence on $Q^B$. 

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Lemma 4 Firm A’s best response \( f^A(Q^B) \) is unique. Let \( \overline{R} = R^iK^i \) for \( i \in \{A, B\} \).

1. If \( Q^B \geq d_H \), then \( f^A(Q^B) \) is constant in \( Q^B \) and satisfies

\[
f^A(Q^B) = \begin{cases} 
   d_L, & \frac{C_A}{C_B} \leq t^A_L \exp(\overline{R}^A d_L) \\
   \frac{1}{R} \ln \left( \frac{C_A}{C_B} \frac{1}{R^A t^A_L} \right), & t^A_L \exp(\overline{R}^A d_L) \leq \frac{C_A}{C_B} \leq t^A_L \exp(\overline{R}^A d_H) \\
   d_H, & t^A_L \exp(\overline{R}^A d_H) \leq \frac{C_A}{C_B} 
\end{cases}
\]

where \( t^A_L \equiv q_L \exp \left( -\overline{R}^A d_L \right) / (1 - q_L) \).

2. If \( d_L \leq Q^B \leq d_H \), then \( f^A(Q^B) \) satisfies

\[
f^A(Q^B) = \begin{cases} 
   d^A_{LL}, & \frac{C_A}{C_B} \leq t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \\
   \frac{1}{R} \ln \left( \frac{C_A}{C_B} \frac{1}{R^A t^A_{LL}} \right), & t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \leq \frac{C_A}{C_B} \leq t^A_{LL} \exp(\overline{R}^A d^A_{LL}) \\
   d^A_{HL}, & t^A_{HL} \exp(\overline{R}^A d^A_{HL}) \leq \frac{C_A}{C_B} \leq t^A_{HL} \exp(\overline{R}^A d^A_{HL}) \\
   \frac{1}{R} \ln \left( \frac{C_A}{C_B} \frac{1}{R^A t^A_{HL}} \right), & t^A_{HL} \exp(\overline{R}^A d^A_{HL}) \leq \frac{C_A}{C_B} \leq t^A_{HL} \exp(\overline{R}^A d^A_{HL}) \\
   d^A_{HH}, & t^A_{HL} \exp(\overline{R}^A d^A_{HH}) \leq \frac{C_A}{C_B} 
\end{cases}
\]

where the demand points are

\[
d^A_{LL} = d_L < d^A_{HL} = d_L + b^A(d_H - Q^B) < d^A_{HH} = d_H < d^A_{HH} = d_H + b^A(d_H - Q^B),
\]

the thresholds \( t^A_{LL} \), \( t^A_{HL} \) and \( t^A_{HL} \) are

\[
t^A_k \equiv \sum_{x, y \in \{L, H\}} q_x q_y \exp(-\overline{R}^A d^A_{xy}) \cdot 1 \{ d^A_{xy} \leq d^A_k \} / \left( 1 - \sum_{x, y \in \{L, H\}} q_x q_y \cdot 1 \{ d^A_{xy} \leq d^A_k \} \right), k \in \{LL, LH, HL\},
\]

and \( t^A_{LL} \) is constant in \( Q^B \), whereas \( t^A_{LL} \) and \( t^A_{HL} \) increase in \( Q^B \).

Figure 1 shows how the primary distribution and the spillover fractions partition the order space. We discuss the figure from firm A’s perspective. If firm B orders at least its maximum primary demand, \( Q^B \geq d_H \), then firm A does not get any spillover demand. It adopts a monopoly strategy based only on its primary demand distribution with mass points \( d_L \) and \( d_H \). However, if firm B orders strictly less than its maximum primary demand, i.e. \( Q^B \in [d_L, d_H] \), then firm A gets \( b^A(d_H - Q^B) \) units of spillover demand from its competitor with probability \( q_H \). For illustration, suppose that firm B orders the quantity indicated by the dashed horizontal line. As a result firm A’s total demand has four possible realizations: low primary demand \( d_L \), shown as point a; low
primary plus spillover demand, \( d_L + b^A(d_H - Q^B) \), point b; high primary demand \( d_H \), point c; and high primary plus spillover demand, \( d_H + b^A(d_H - Q^B) \), point d.

Figure 1: Primary demand distribution and spillover fractions partition the order space.

The impact of a small change in firm B’s order quantity on firm A’s marginal underage and overage risks is sensitive to how much firm A orders. If firm A orders less than low primary plus spillover demand, \( Q^A < d_L + b^A(d_H - Q^B) \), then A sells out if either firm experiences high demand. In this case a small change in \( Q^B \) has no impact on A’s profitability since it does not alter the amount of spillover demand that A can satisfy. The profitability of firm A only depends on \( Q^B \) if it orders at least low primary plus spillover demand, \( Q^A \geq d_L + b^A(d_H - Q^B) \). In this case, if firm A has low primary demand then it leftover inventory \((Q^A - d_L)\) exceeds her potential spillover demand. If \( Q^A < d_H \), then A sells out only if its own primary demand is high, so it benefits from spillover demand only if its own primary demand is low. If firm A orders more than high primary demand \( Q^A \geq d_H \) then it sells out only if both firms’ primary demands are high, so it benefits from spillover demand regardless of her own demand realization. Note that if \( Q^A \leq d_H + b^A(d_H - Q^B) \) then the impact of a marginal change in \( Q^B \) on A’s profitability only comes from higher sales under low primary demand: if firm A has high primary demand then its overstock is \( Q^A - d_H \) without spillover demand and zero with spillover demand, independent of \( Q^B \).

Remark and assumption. If \( b^A = b^B = 1 \), the equilibrium is unique if \( Q^{A*} + Q^{B*} \neq d_L + d_H \). Otherwise there may exist a continuum of equilibria along the line \( Q^A + Q^B = d_L + d_H \). For simplicity we henceforth assume that \( b^A, b^B < 1 \).

The impact of risk aversion on the equilibrium orders depends on the location of the initial equilibrium, relative to the order space partition portrayed in Figure 1. As a stepping stone for
the analysis of the risk aversion impact, Proposition 2 categorizes the possible equilibria into 17
distinct cases, each corresponding to a particular subset of the order space.

**Proposition 2** The location of the equilibrium orders \((Q^A, Q^B)\) can be categorized into the fol-
lowing cases. For concreteness, let \(i = A\) and \(-i = B\).

<table>
<thead>
<tr>
<th>Case</th>
<th>Firm A Equilibrium Order (Q^A)</th>
<th>Conditions</th>
<th>Firm B Equilibrium Order (Q^B)</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(d_{LH}^A (Q^B))</td>
<td>(Q_{LL}^A (Q^B) \leq d_{LH}^A (Q^B) &lt; Q_{LL}^A)</td>
<td>(Q_{LL}^B)</td>
<td>(d_{L} &lt; Q_{LL}^B \leq d_{LH}^A (Q^A))</td>
</tr>
<tr>
<td>2</td>
<td>(d_{LH}^A (Q^B))</td>
<td>(Q_{LL}^A (Q^B) \leq d_{LH}^A (Q^B) &lt; Q_{LL}^A)</td>
<td>(Q_{LL}^B (Q^A))</td>
<td>(d_{LH}^A (Q^A) &lt; Q_{LL}^B (Q^A) &lt; d_{H})</td>
</tr>
<tr>
<td>3</td>
<td>(Q_{HL}^A (Q^B))</td>
<td>(d_{LH}^A (Q^B) &lt; Q_{HL}^A (Q^B) \leq d_{H})</td>
<td>(Q_{LL}^B (Q^A))</td>
<td>(d_{LH}^A (Q^A) &lt; Q_{LL}^B (Q^A) \leq d_{H})</td>
</tr>
<tr>
<td>4</td>
<td>(Q_{HL}^A (Q^B))</td>
<td>(d_{LH}^A (Q^B) &lt; Q_{HL}^A (Q^B) \leq d_{H})</td>
<td>(Q_{LL}^B (Q^A))</td>
<td>(d_{LH}^A (Q^A) &lt; Q_{LL}^B (Q^A) \leq d_{H})</td>
</tr>
<tr>
<td>5</td>
<td>(Q_{HL}^A (Q^B))</td>
<td>(d_{H} \leq Q_{HL}^A (Q^B) &lt; d_{LH}^A (Q^B))</td>
<td>(Q_{M}^B)</td>
<td>(d_{L} &lt; Q_{M}^B &lt; d_{H})</td>
</tr>
<tr>
<td>6</td>
<td>(d_{HH}^A (Q^B))</td>
<td>(d_{L} \leq Q_{HH}^A (Q^B) &lt; d_{LH}^A (Q^B))</td>
<td>(Q_{M}^B)</td>
<td>(d_{L} &lt; Q_{M}^B &lt; d_{H})</td>
</tr>
<tr>
<td>7</td>
<td>(d_{HH}^A (Q^B))</td>
<td>(d_{L} \leq Q_{HH}^A (Q^B) &lt; d_{LH}^A (Q^B))</td>
<td>(Q_{M}^B)</td>
<td>(d_{L} \leq Q_{M}^B \leq d_{L})</td>
</tr>
<tr>
<td>8</td>
<td>(Q_{HL}^A (Q^B))</td>
<td>(d_{H} \leq Q_{HL}^A (Q^B) \leq d_{LH}^A (Q^B))</td>
<td>(d_{L})</td>
<td>(Q_{M}^B \leq d_{L})</td>
</tr>
<tr>
<td>9</td>
<td>(d_{H})</td>
<td>(d_{H} \leq Q_{M}^A)</td>
<td>(d_{H})</td>
<td>(d_{H} \leq Q_{M}^B)</td>
</tr>
<tr>
<td>10</td>
<td>(d_{H})</td>
<td>(Q_{HL}^A (Q^B) \leq d_{H} &lt; Q_{HL}^A (Q^B))</td>
<td>(Q_{M}^B)</td>
<td>(d_{L} &lt; Q_{M}^B &lt; d_{H})</td>
</tr>
<tr>
<td>11</td>
<td>(d_{H})</td>
<td>(Q_{HL}^A (Q^B) \leq d_{H} &lt; Q_{HL}^A (Q^B))</td>
<td>(d_{L})</td>
<td>(Q_{M}^B \leq d_{L})</td>
</tr>
<tr>
<td>12</td>
<td>(Q_{HL}^A (Q^B))</td>
<td>(d_{LH}^A (Q^B) \leq Q_{HL}^A (Q^B) \leq d_{H})</td>
<td>(d_{L})</td>
<td>(Q_{LH}^B \leq d_{L})</td>
</tr>
<tr>
<td>13</td>
<td>(d_{LH}^A (Q^B))</td>
<td>(Q_{HL}^A (Q^B) \leq d_{H} &lt; Q_{HL}^A (Q^B))</td>
<td>(d_{LH}^A (Q^A))</td>
<td>(d_{LH}^A (Q^A) \leq Q_{LL}^B (Q^A) &lt; Q_{LL}^B)</td>
</tr>
<tr>
<td>14</td>
<td>(d_{LH}^A (Q^B))</td>
<td>(Q_{HL}^A \leq d_{LH}^A &lt; Q_{LL}^A)</td>
<td>(d_{L})</td>
<td>(Q_{LH}^B \leq d_{L})</td>
</tr>
<tr>
<td>15</td>
<td>(Q_{LL}^A)</td>
<td>(d_{LH}^A \leq d_{LH}^A \leq d_{LH}^A)</td>
<td>(Q_{LH}^B)</td>
<td>(d_{L} \leq Q_{LH}^B \leq d_{LH}^A (Q^A))</td>
</tr>
<tr>
<td>16</td>
<td>(Q_{LL}^A)</td>
<td>(d_{LH}^A \leq d_{LH}^A \leq d_{LH}^A)</td>
<td>(d_{L})</td>
<td>(Q_{LH}^B \leq d_{L})</td>
</tr>
<tr>
<td>17</td>
<td>(d_{L})</td>
<td>(Q_{LL}^A \leq d_{L})</td>
<td>(d_{L})</td>
<td>(Q_{LH}^B \leq d_{L})</td>
</tr>
</tbody>
</table>

The demand points satisfy

\[
d_{L} < d_{LH}^i (Q) = d_{L} + b^i (d_{H} - Q) < d_{H} < d_{H}^i (Q) = d_{H} + b^i (d_{H} - Q) \text{ for } Q \in [d_{L}, d_{H}], \tag{21}
\]
and the functions $Q^i_M$, $Q^i_{LL}$, $Q^i_{LH}(Q)$, and $Q^i_{HL}(Q)$ are defined as

\begin{align}
Q^i_M &\triangleq \frac{1}{R} \ln \left( \frac{C^i_u}{C^i_o} \frac{1 - q_L}{q_L \exp(-R d_L)} \right), \\
Q^i_{LL} &\triangleq \frac{1}{R} \ln \left( \frac{C^i_u}{C^i_o} \frac{1 - q_{LL}}{q_{LL} \exp(-R d_L)} \right), \\
Q^i_{LH}(Q) &\triangleq \frac{1}{R} \ln \left( \frac{C^i_u}{C^i_o} \frac{1 - q_{LL} - q_{LH}}{q_{LL} \exp(-R d_L) + q_{LH} \exp(-R d_{LH}(Q))} \right), \\
Q^i_{HL}(Q) &\triangleq \frac{1}{R} \ln \left( \frac{C^i_u}{C^i_o} \frac{1 - q_{LL} - q_{LH} - q_{HL}}{q_{LL} \exp(-R d_L) + q_{LH} \exp(-R d_{LH}(Q)) + q_{HL} \exp(-R d_H)} \right).
\end{align}

Note that $Q^i_{LL} > Q^i_{LH}(Q) > Q^i_{HL}(Q)$ and $Q^i_{HL}(d_H) = Q^i_M$.

For concreteness Figure 2 illustrates these cases from the perspective of firm $A$, and the Proposition specifies the cases accordingly. Proposition 2 serves two purposes for our analysis.

First, it helps us identify the cases in which higher risk aversion may yield a strictly higher equilibrium order for one of the firms: they are the cases 1-6, as we show in Section 5.2. In all other cases, higher risk aversion implies (weakly) lower equilibrium orders for both firms.

Second, Proposition 2 yields specific parameter conditions that must hold in each equilibrium case. Combined with the forthcoming analysis, these conditions allow us to systematically analyze and identify the impact of higher risk aversion as a function of the model parameters. For illustration, consider Case 1 of Proposition 2. Firm $A$ orders its low primary plus spillover demand, and
firm B orders more than low primary demand but less than low primary plus spillover demand. Substituting from (21) and (23) yields the equilibrium order quantities in closed form:

\[ Q^A = \frac{d_{LL}^A}{R^B} (Q_{LL}^B) = d_L + b^A \left( d_H - \frac{1}{R^B} \ln \left( \frac{C_u^B}{C_o^B} \right) \right) \]

and

\[ Q^B = Q_{LL}^B = \frac{1}{R^B} \ln \left( \frac{C_u^B}{C_o^B} \right) \]

Substituting these quantities into the equilibrium conditions yields conditions that only involve the problem parameters. All other cases, except for Cases 2 and 3, also yield closed-form expressions for the equilibrium order quantities. The equilibrium equations for Cases 2 and 3 are easily solved numerically.

The following intuitive equilibrium properties are straightforward from Lemmas 4-2.

**Corollary 2** If both firms are identical except for:

1. Their profitability, measured by the ratio \( C_u^i / C_o^i \), while \( C_u^i + C_o^i = C_u^{-i} + C_o^{-i} \), then the equilibrium order of the firm with the higher ratio (weakly) exceeds that of its rival.

2. Their risk aversion rate \( R^i \), then the equilibrium order of the less risk-averse firm (weakly) exceeds that of its rival.

3. Their spillover fraction \( b^i \), then the equilibrium order of the firm with the higher spillover fraction (weakly) exceeds that of its rival.

### 5.2 Impact of Risk Aversion on Equilibrium Orders

We turn to the main question of this paper: how does the order quantity equilibrium of risk-averse duopoly firms change in response to a change in their risk aversion?

Theorem 2 identifies, for our demand model with i.i.d. binary primary demand and deterministic spillover fractions, which equilibrium cases of Proposition 2 satisfy the conditions of Theorem 1, and under what additional conditions. This analysis shows how the impact of increasing risk aversion depends on the location of the initial equilibrium, and it yields specific conditions that can be evaluated numerically to determine the set of model parameters for which (7) holds.

We illustrate and discuss the results of Theorem 2 with numerical examples for three scenarios of asymmetric firms that differ in exactly one of the following attributes: (i) the profitability of their product, measured by their under- to overstocking cost ratio \( C_u^i / C_o^i \), (ii) their risk aversion parameter \( R^i \), and (iii) their spillover demand fraction \( b^i \).
In our model framework, the magnitudes of the own risk aversion and demand substitution effects are closely linked to the location of the initial equilibrium relative to the order space partition. By exploiting these relationships, we can narrow down the set of all possible equilibrium cases identified in Proposition 2 to only a handful of relevant ones.

**Theorem 2** Fix $R > 0$ and the corresponding equilibrium $Q^*(R)$. For concreteness, let $i = A$ and $-i = B$. The equilibrium order of firm $A$ increases as both firms become more risk averse,

$$\frac{\partial Q^*_A(R)}{\partial R^A} + \frac{\partial Q^*_B(R)}{\partial R^B} > 0,$$

if and only if $Q^*(R)$ satisfies one of Cases 1-6 of Proposition 2 and further conditions in Cases 3-5.

1. Firm $A$ orders minimum demand plus spillover, firm $B$ between minimum demand and minimum demand plus spillover:

$$Q^A_{LH}(Q^B) \leq Q_A^* = d_{LH}^A(Q^B) < Q^A_{LL} < d_L \leq Q^B = Q^B_{LL} \leq d_B^B(Q^A).$$

2. Firm $A$ orders minimum demand plus spillover, firm $B$ between minimum demand plus spillover and maximum primary demand:

$$Q^A_{LH}(Q^B) \leq Q_A^* = d_{LH}^A(Q^B) < Q^A_{LL} < d_L \leq Q^B = Q^B_{LH}(Q^A) < d_H.$$  

3. Both firms order between minimum demand plus spillover and maximum primary demand:

$$d_{LH}^i(Q^{-i}) < Q^i_0 = Q^i_{LH}(Q^{-i}) < d_H, \quad i \in \{A, B\},$$

and moreover,

$$\frac{\partial Q^B_{LH}(Q^A, R^B) - \partial Q^A_{LH}(Q^B, R^A)}{\partial R^B} > -\frac{\partial Q^B_{LH}(Q^A, R^A)}{\partial R^A}. \tag{27}$$

4. Firm $A$ orders between minimum demand plus spillover and maximum primary demand, firm $B$ between minimum demand and minimum demand plus spillover:

$$d_{LH}^A(Q^B) < Q_A^* = Q^A_{LH}(Q^B) \leq d_H \quad \text{and} \quad d_L \leq Q^B = Q^B_{LL} \leq d_B^B(Q^A),$$

and moreover,

$$Q^B_{LL}(R^B) \frac{\partial Q^A_{LH}(Q^B, R^A)}{\partial Q^B} > -\frac{\partial Q^A_{LH}(Q^B, R^A)}{\partial R^A}. \tag{28}$$
5. Firm A orders between maximum primary demand and maximum demand, firm B between minimum and maximum primary demand:

\[ d_H \leq Q^A = Q^A_{HL} (Q^{B*}) \leq d_{HH}^A (Q^{B*}) \] and \( d_L < Q^{B*} = Q^B_M < d_H \),

and moreover,

\[ Q^B_M (R^B) \frac{\partial Q^A_{HL}(Q^{B*}, R^A)}{\partial Q^B} > - \frac{\partial Q^A_{HL}(Q^{B*}, R^A)}{\partial R^A}. \] (29)

6. Firm A orders maximum primary demand plus spillover, firm B between minimum and maximum primary demand:

\[ Q^A = d_{HH}^A (Q^{B*}) < Q^A_{HL} (Q^{B*}) \] and \( d_L < Q^{B*} = Q^B_M < d_H \).

Remark. Except for the equilibrium conditions for Case 3, all the conditions in the Theorem can be translated into conditions that only involve the model parameters. See Proposition 2 for the equilibrium conditions, and the proof of 2 for the additional conditions (27)-(29).

Figure 3 illustrates the equilibrium cases identified in Theorem 2 for firm A.

In Cases 1, 2, and 6, firm A always increases its order in response to higher risk aversion levels. In these cases, firm A’s initial equilibrium equals low or high primary demand plus spillover demand, and it is locally insensitive to higher risk aversion. Its own risk aversion effect is therefore zero, whereas the demand substitution effect is strictly positive.

In Cases 3, 4, and 5, firm A may, but need not, increase its equilibrium order. Firm A increases its order if and only if the appropriate additional condition from (27)-(29) holds, which measure the magnitudes of the own risk aversion and the demand substitution effects for each case.

By Theorem 2, neither firm increases its equilibrium order in response to higher risk aversion levels in the remaining eleven equilibrium cases of Proposition 2. We briefly explain why.

In seven of these cases, one firm, call it \( i \), orders the minimum demand, \( d_L \). At this level, firm \( i \) has no overstocking risk. An increase in spillover demand from its rival has no impact on firm \( i \)’s over- and understocking risks, so it does not change firm \( i \)’s order. Since firm \( i \) will not order less than \( d_L \), its rival does not increase its order in response to higher risk aversion.

In Case 9, firms are symmetric, and Part 1 of Corollary 1 applies.

In Case 10, firm A initially orders its maximum primary demand, and it is locally insensitive to higher risk aversion. Therefore, it does not change its order in response to higher risk aversion, while the order of its rival decreases.

In Cases 13 and 15, both firms initially order less than or equal to their minimum demand plus spillover demand. In these cases, the order quantities of both firms are so low that their spillover
Figure 3: Theorem 2. Equilibrium cases in which one of the firm always increases (cases 1,2,6) or may increase (cases 3,4,5) its order in response to higher risk aversion levels.

demand is enough for them to sell out. A small reduction in one firm’s order quantity therefore does not alter the overstocking risks of its rival, and the demand substitution effect is zero. Each firm maintains or reduces its order, depending on whether its own risk aversion effect is zero or not.

In the next Sections 5.3-5.5 we illustrate and discuss the cases identified in Theorem 2 with numerical examples for three scenarios of firm asymmetries, considering in turn firms that differ in (i) the profitability of their product, measured by their under- to overstocking cost ratio $\frac{C_u^i}{C_o^i}$, (ii) their risk aversion parameter $R^i$, and (iii) their spillover demand fraction $b^i$.

5.3 Different Profitability ($C_u^i/C_o^i$)

A key measure in the newsvendor model is the ratio of under- to overstocking cost. In this Section we consider firms that are identical, except for the profitability of their products, measured by their $C_u^i/C_o^i$ ratios. However, we assume that the difference between price and salvage value is the same for both firms: $C_u^i + C_o^i = K = K^i$. In settings with equal prices, the firm with the higher $C_u^i/C_o^i$ ratio has the lower unit cost. In settings with equal unit costs, the firm with the higher $C_u^i/C_o^i$ ratio commands a higher price and salvage value. Our results show that the firm that responds with a higher equilibrium order to an increase in risk aversion can be the one with the higher or the one with the lower under- to overstocking cost ratio. We identify which case applies under what conditions and explain why.
Corollary 3 Suppose that both firms are identical except for their profitability, measured by their $C_u^i/C_o^i$ ratios, but $C_u^i + C_o^i = K^i = K^{-i}$. By Part 1. of Corollary 2, the equilibrium order of the firm with the higher ratio (weakly) exceeds that of its rival.

1. In the following cases of Theorem 2, as both firms become more risk averse, the more profitable firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes larger.

(a) Case 1. The more profitable firm orders minimum demand plus spillover, its rival orders between its minimum demand and minimum demand plus spillover.

(b) Case 4. The more profitable firm orders between minimum demand plus spillover and maximum primary demand, its rival orders between minimum demand and minimum demand plus spillover.

(c) Case 6. The more profitable firm orders maximum demand plus spillover, its rival orders between minimum and maximum primary demand.

2. In the following cases of Theorem 2, as both firms become more risk averse, the less profitable firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes smaller.

(a) Case 2. The less profitable firm orders minimum demand plus spillover, its rival orders between minimum demand plus spillover and maximum primary demand.

(b) Case 3. Both firms order between minimum demand plus spillover and maximum primary demand.

3. Furthermore, Case 5 of Theorem 2 never holds.

Figure 4 illustrates Theorem 2 and Corollary 3. Fixing all the model parameters, except for the $C_u^i/C_o^i$ ratios, at the values indicated in the Figure, it identifies the pairs $(C_u^A/C_o^A, C_u^B/C_o^B)$ that yield one of the cases of Theorem 2. The shaded areas in Figure 4 illustrate the regions in $(C_u^A/C_o^A, C_u^B/C_o^B)$-space where one of the firms will increase its order quantity as both firms become more risk averse. In the rest of the space, neither of the firms will increase their order quantity as both firms become more risk averse. As shown in Figure 4(a), in some cases it is the more profitable firm which increases its order quantity, in other cases it is the less profitable one. Since these regions are symmetric about the diagonal, we focus in our discussion on the cases where
firm A increases its order quantity hereafter. Figure 4(b) represents these cases. For the model parameters in this example the conditions of Case 4 of Theorem 2 are not satisfied. The equilibrium Case 5 does not appear in the Figure, consistent with Corollary 3. This leaves equilibrium Cases 1 and 6, in which firm A has the higher $C_u^i/C_o^i$ ratio, and Cases 2 and 3, where firm A has the lower $C_u^i/C_o^i$ ratio.

In Cases 1, 2 and 6, firm A’s initial equilibrium order is on a demand point that depends on its rival spillover, which has two implications. First, firm A’s own risk aversion effect is zero. Second, its best response strictly increases if her rival’s order decreases: the resulting larger spillover demand increases firm A’s expected marginal utility from ordering an extra unit since this additional unit is more likely to sell than under lower spillover demand. Since firm B’s own risk aversion effect is strictly positive in each of these Cases, the demand substitution effect is positive for firm A and it orders more in response to higher risk aversion. The Cases 1, 2 and 6 differ in the relative magnitudes of the firms’ $C_u^i/C_o^i$ ratios. In Case 6, firm A is much more profitable than firm B, and its initial order equals maximum primary demand plus spillover demand. At the representative point $a$, the ratios are $C_u^A/C_o^A = 4$ and $C_u^B/C_o^B = 1$. In Cases 1 and 2, firm A is much less profitable. E.g., $C_u^A/C_o^A = 0.5$ at the representative points $b$ and $c$. The same marginal effects are at play at both points, causing firm A to increase its order in response to higher risk aversion. The only difference between the points is that firm B is the less profitable firm at point $b$ and the more profitable one at point $c$.

![Figure 4: Illustration of Theorem 2. Impact of risk aversion depending on understocking to overstocking ratios $C_u/C_o$ of both firms. Parameters: $K^i = 0.5, R^i = 0.3, b^i = 0.5, q_L = 0.4, d_L = 5, d_H = 10$.](image)

In the equilibrium Case 3, both firms order more than minimum demand plus spillover but less than maximum primary demand. Unlike in the preceding Cases, both firms’ own risk aversion
effect is nonzero. As discussed above, only the firm with the strictly lower own risk aversion effect can increase its equilibrium order in response to higher risk aversion. It can be shown that, holding other factors fixed, the magnitude of a firm’s own risk aversion effect is increasing in its initial order quantity. Intuitively, the larger the order quantity, the larger a firm’s risk exposure and the larger the marginal increase in this risk in response to higher risk aversion. Since the firm with the lower $C_u^i/C_o^i$ ratio orders less than its rival (if firms are otherwise symmetric), it faces a smaller own risk aversion effect. It is therefore the firm that increases its order in response to higher risk aversion. In Figure 4(b), this is the Case for firm $B$ for values of $(C_u^A/C_o^A, C_u^B/C_o^B)$ in region 3.

5.4 Different Risk Aversion ($R'$)

Corollary 4 Suppose that both firms are identical, except that they have different initial risk aversion. By Part 2. of Corollary 2, the equilibrium order of the initially less risk-averse firm (weakly) exceeds that of its rival.

1. In the following Cases of Theorem 2, as both firms become more risk averse, the initially less risk averse firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes larger.

   (a) Case 1. The initially less risk averse firm orders minimum demand plus spillover, its rival orders between its minimum demand and minimum demand plus spillover.

   (b) Case 4. The initially less risk averse firm orders between minimum demand plus spillover and maximum primary demand, its rival orders between minimum demand and minimum demand plus spillover.

   (c) Case 6. The initially less risk averse firm orders maximum demand plus spillover, its rival orders between minimum and maximum primary demand.

2. In the following Cases of Theorem 2, as both firms become more risk averse, the initially more risk averse firm increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes smaller.

   (a) Case 2. The initially more risk averse firm orders minimum demand plus spillover, its rival orders between minimum demand plus spillover and maximum primary demand.

   (b) Case 3. Both firms order between minimum demand plus spillover and maximum primary demand.
3. Furthermore, Case 5 of Theorem 2 never holds.

The cases in which the initially less (more) risk averse firm increases its order quantity are comparable to the cases in which the more (less) profitable firm increases its order quantity. Refer to Figure 5 which illustrates the Corollary numerically. As in Figure 4, for the model parameters in this example the conditions of Case 4 of Theorem 2 are not satisfied. As Figure 5 shows, there is a wide range of initial risk aversion parameters such that one of the firms increases its equilibrium order quantity as both firms become more risk averse. This range is highly dependent on the firms’ understocking to overstocking ratios. When $C_i^u/C_o^u = 0.6$, in Figure 5(a), if either firm increases its order quantity as both firms become more risk averse, it is the less risk averse firm. However, when $C_i^u/C_o^u = 2.5$, in Figure 5(b), it can be either the more or the less risk averse firm that increases its order quantity as both firms become more risk averse.

Further note that certain equilibrium cases only emerge at higher profitability ratios. For example, fix $R^A = 0.7$ and $R^B = 3$. Under low profitability ratios for both firms, it is the less risk averse firm (A) that increases its order as both firms become more risk averse; see Figure 5(a). Under low profitability, the initial equilibrium orders are relatively small and the corresponding equilibrium meets the conditions of Case 1 of Theorem 2. By contrast, under relatively large profitability ratio for firms, it is the more risk averse firm (B) that increases its order quantity as both firms become more risk averse; see Figure 5(b). Under high profitability, the initial equilibrium orders are higher and the corresponding equilibrium has the characteristics of Case 3 of Theorem 2. In this case, both firms order between minimum demand plus spillover and high demand, and their own risk aversion effects are nonzero. Since the more risk averse firm B orders less initially, its own risk aversion effect is smaller than that of its rival.

Observe in Figure 5(b) that for fixed initial risk aversion parameter of firm A ($R^A = 4$), as firm B’s initial risk aversion parameter varies from very low to very high, the relative sensitivity of the firms in response to higher risk aversion changes. When firm B’s initial risk aversion rate is very low or quite high, lower than 0.2 or in the 2.3-2.8 range, then it is firm B that increases its order quantity as both firms become more risk averse (the equilibria for very low $R^B$ are instances of equilibrium Case 6 of Theorem 2, whereby firm B orders high demand plus spillover; the equilibria for values of $R^B$ in the 2.3-2.8 range are instances of equilibrium Case 1 of Theorem 2, whereby firm B orders minimum demand plus spillover.) However, for $R^B$ in some intermediate range, 0.5-2.3, it is firm A that increases its order quantity as both firms become more risk averse (these are instances of equilibrium Cases 2 and 3 of Theorem 2.
Figure 5: Illustration of Theorem 2. Impact of risk aversion depending on initial risk aversion levels $R^i$ of both firms. Parameters: $K^i = 0.5, b^i = 0.5, q_L = 0.4, d_L = 5, D_H = 10$.

5.5 Different Spillover Fractions ($b^i$)

Corollary 5 Suppose that both firms are identical, except for their spillover fractions. By Part 3. of Corollary 2, the equilibrium order of the firm with the higher spillover fraction (weakly) exceeds that of its rival.

1. In the following cases of Theorem 2, as both firms become more risk averse, the firm with the lower spillover fraction increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes smaller.

   (a) Case 1. The firm with lower spillover fraction orders minimum demand plus spillover, its rival orders between its minimum demand and minimum demand plus spillover.

   (b) Case 4. The firm with lower spillover fraction orders between minimum demand plus spillover and maximum primary demand, its rival orders between minimum demand and minimum demand plus spillover.

2. In Case 2 of Theorem 2, as both firms become more risk averse, the firm with the higher spillover fraction increases its equilibrium order, and the difference between the firms’ equilibrium orders becomes larger. The firm with the higher spillover fraction orders minimum demand plus spillover, its rival orders between minimum demand plus spillover and maximum primary demand.

3. Furthermore, Cases 5 and 6 of Theorem 2 never hold.

Figure 6 shows that for a range of spillover fractions, one of the firm may increase its order quantity as both become more risk averse, but this range is highly dependent on profitability ratio.
Note that, as in the previous examples, for the model parameters in this example the conditions of Case 4 of Theorem 2 are not satisfied. For example, when $b^A = 0.7$, and $b^B = 0.3$, if the firms are relatively less profitable ($C^i_o/C^i_o = 0.25$), it is the firm with lower spillover rate (Firm B) that increases its order quantity as both firms become more risk averse. This case refers to Part 1 of Theorem 2. However if the firms are more profitable ($C^i_o/C^i_o = 0.75$) the firm with higher spillover rate (Firm A) increases its order quantity as both firms become more risk averse. This case refers to Part 2 of Theorem 2.

![Figure 6: Illustration of Theorem 2. Impact of risk aversion depending on spillover fractions $b^i$ of both firms. Parameters $K^i = 0.5$, $R^i = 0.3$, $q_L = 0.4$, $d_L = 5$, $d_H = 10$.](image)

Unlike in the cases where firms differ in their risk aversion parameters ($R^i$), varying the spillover rate ($b^i$) has a less significant effect and it highly depends on the profitability ratio $C^i_o/C^i_o$. When both firms are identical except for their spillover fraction, a high profitability ratio leads to equilibrium for both firms, independent of their individual spillover fractions. Similarly, low profitability leads to low equilibrium orders. As a result, some of the equilibrium cases do not emerge under asymmetric spillover fractions. For example, it cannot be an equilibrium for one firm, say firm A, to order maximum primary demand plus spillover and the firm B to order between minimum demand and maximum primary demand. To order maximum primary demand plus spillover, firm A should be so profitable that it orders more than its own high demand when it adopts monopoly strategy. Since we assume same profitability in this section, firm B must be equally profitable, so that the equilibrium orders of both firms would equal their maximum primary demand. In these cases, one can argue that the profitability is the main drivers of the equilibrium orders, and spillover demand results in limited deviation on top of that.
6 Sensitivity Analysis: Primary Demand Correlation

The analysis has so far assumed uncorrelated primary demands. However, since the products offered by the firms are partially substitutable, their primary demands might also be correlated. In this section we relax the independent demand distribution assumption of Section 5 to incorporate primary demand correlation.

6.1 Model and Equilibrium Characterization

We model correlation by using conditional probability: Given that firm $A$ experiences low primary demand $d_L$, firm $B$ will experience the same primary demand with probability $\theta_1$, and given that firm $A$ experiences high primary demand $d_H$, firm $B$ will also do so with probability $\theta_2$. The resulting p.d.f. of firm $A$ total demand $D^A(Q^B)$ is

\[
P(D^A(Q^B) = d) = \begin{cases} 
q_{LL} \triangleq q_L v_1, & d = d_{LL}^A(Q^B) \triangleq d_L \\
q_{LH} \triangleq q_L (1 - v_1), & d = d_{LH}^A(Q^B) \triangleq d_L + b^A[d_H - Q^B]^+ \\
q_{HL} \triangleq q_H (1 - v_2) = (1 - q_L)(1 - v_2), & d = d_{HL}^A(Q^B) \triangleq d_H \\
q_{HH} \triangleq q_H v_2 = (1 - q_L)v_2, & d = d_{HH}^A(Q^B) \triangleq d_H + b^A[d_H - Q^B]^+ 
\end{cases}.
\]

This p.d.f. is structurally equivalent to (14) in the uncorrelated case, it only differs in the probabilities of the various outcomes. If we set $\theta_1 = \theta_2 = 1$, then (30) specializes to (14). Hence, all the previous analysis and results in Section 5 can be replicated for this case by using appropriate probabilities. We focus on two extreme cases: perfect positive and perfect negative correlation.

Note that as pointed out in Section 2, our total demand model is very generic and it can accommodate different initial demand allocation rules. For example, in our model in Section 5, the initial demand allocation is similar in nature to that in Parlar (1988). It is also the deterministic counterpart of the probabilistic splitting rule "Independent Random Demands" in Lippman and McCardle (1997). By incorporating primary demand correlation in our model, we can represent different initial allocation rules that are discussed in Lippman and McCardle (1997).

**Perfect positive primary demand correlation.** In this case, $v_1 = v_2 = 1$, and (30) specializes to

\[
P(D^A(Q^B) = d) = \begin{cases} 
\quad d = d_{LL}^A(Q^B) = d_L & \text{with probability } q_L \\
\quad d = d_{HH}^A(Q^B) = d_H + b^A[d_H - Q^B]^+ & \text{with probability } (1 - q_L) 
\end{cases}.
\]
As a result, the best response function of firm A is:

\[
Q^A = \begin{cases} 
  d_{LL}^A = d_L \\ 
  d_L + \frac{1}{R} \ln \left[ \frac{C_A^u}{C_A^\delta} \left( \frac{1-qL}{qL} \right) \right] & t_{LL}^A < \frac{C_A^u}{C_A^\delta} \\
  d_{HH}^A = d_H + b^A [d_H - Q^B]^+ & t_{LL}^A e^{R^A(d_{HH}^A(Q^B) - d_{LL}^A(Q^B))} < \frac{C_A^u}{C_A^\delta}
\end{cases}
\]

where

\[
t_{LL}^A = \frac{qL}{1-qL}.
\]

**Perfect negative primary demand correlation.** In this case, \( v_1 = v_2 = 0 \), and (30) specializes to

\[
P(D^A(Q^B) = d) = \begin{cases} 
  d = d_{LH} = d_L + b^A[d_H - Q^B]^+ & \text{with probability } qL \\
  d = d_{HL} = d_H & \text{with probability } (1-qL)
\end{cases}
\]

As a result, the best response function of firm A is:

\[
Q^A = \begin{cases} 
  d_{LH}^A = d_L + b^A(d_H - Q^B)^+ \\
  d_L + b^A(d_H - Q^B)^+ + \frac{1}{R} \ln \left[ \frac{C_A^u}{C_A^\delta} \left( \frac{1-qL}{qL} \right) \right] & t_{LL}^A < \frac{C_A^u}{C_A^\delta} \\
  d_{HL}^A = d_H & t_{LL}^A e^{R^A(d_{HH}^A(Q^B) - d_{LL}^A(Q^B))} < \frac{C_A^u}{C_A^\delta}
\end{cases}
\]

\[
(33)
\]

### 6.2 Impact of Risk Aversion: Perfect Correlation

We discuss the impact of risk aversion under perfect primary demand correlation.

**Theorem 3** For concreteness, let \( i = A \) and \( j = B \). Suppose the primary demands of the two firms are perfectly positively correlated. As both firms become more risk averse, firm A increases its order quantity if and only if the initial equilibrium is such that firm A orders maximum primary demand plus spillover, firm B orders between minimum demand \((d_L)\) and maximum primary demand \((d_H)\)

\[
Q^A = d_H + b^A(d_H - Q^B) \quad \text{and} \quad Q^B = d_L + \frac{\ln \left[ \frac{C_B^u}{C_B^\delta} \left( \frac{1-qL}{qL} \right) \right]}{R^B}
\]

This is the unique equilibrium if the following conditions hold

\[
\frac{C_A^u}{C_A^\delta} > \frac{qL e \left( (1+b^A)(d_H - d_L) - b^A \ln \left[ \frac{C_B^u}{C_B^\delta} \left( \frac{1-qL}{qL} \right) \right] \right)}{1-qL} = t_{LL}^A e^{R^A(d_{HH}^A(Q^B) - d_{LL}^A(Q^B))},
\]

\[
t_{LL}^B = \frac{qL}{1-qL} \leq \frac{C_B^u}{C_B^\delta} \leq \frac{qL e R_B^B(d_H - d_l)}{1-qL} = t_{LL}^B e^{R^B(d_{HH}^B(Q^A) - d_{LL}^B(Q^A))}
\]

Similar statement, as above, is valid for firm B.

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The case in Theorem 3 corresponds to the equilibrium Case 6 of Proposition 2 and Theorem 2. The other five equilibrium cases do not apply here, due to perfect positive primary demand correlation.

Under perfect positive primary demand correlation, firm A experiences either low primary demand or high primary demand plus potential spillover \((d_H + b^A[d_H - Q_B]^+)\). Therefore, firm A’s order quantity can be its low demand, between its low demand and high demand plus spillover, or equal its high demand plus spillover demand. We investigate the impact of risk aversion under each possible initial equilibrium.

1. If both firms order less than high primary demand \((d_H)\), both of them order the monopoly order quantity. At this quantity, they do not require any spillover demand to sell out, and a change in spillover demand does not change their overstocking cost. As a result, their own risk aversion effect is the only factor that has an impact on both firms order quantity. Therefore both firms reduce their order quantity as both become more risk averse.

2. One of the firms, say firm A, orders more than its high primary demand, but less than high demand plus spillover, and firm B orders less than high demand. In this case also, both firms choose the monopoly order quantity. Although one of the firms orders more than its high primary demand, the current level of spillover demand is enough to sell out, so that more spillover demand does not reduce its overstocking risk. In this case, since primary demands are perfectly correlated, firm A will sell out if it experiences the high demand. Hence, as in the previous case, only the own risk aversion effect has an impact on both firms order quantity, and both firms reduce their order quantity in response to higher risk aversion.

3. If firm A’s competitor (firm B) orders at least high demand \((d_H)\), firm A only experiences the low or high demand as it does not get any spillover demand. Therefore it will adapt monopoly strategy. Furthermore, note that if firm B orders less than the high demand \((d_H)\) and firm A orders between its low demand and high demand plus spillover, firm A’s order quantity will be the the monopoly order quantity (independent of firm B’s order quantity).

As a result, the only case in which firm A’s order quantity depends on that of its competitor is the case in which firm A orders maximum demand plus spillover. In all other cases firm A’s order quantity is independent of its competitor’s order quantity. Therefore, in those cases, as both firms become more risk averse, "own risk aversion increment" is the only factor that has an impact on firm A’s order quantity. Hence firm A reduces its order quantity. However, when firm A’s order quantity is maximum demand plus spillover, the impact of spillover demand is influential. In this case firm A should be quite profitable (and/or significantly less risk averse) to order the maximum demand plus spillover. Such a high order quantity \((Q^A > d_H)\) results in no spillover
demand for firm B. On the other hand firm B’s profitability should be in the intermediate range (and/or relatively more risk averse), to have an initial order quantity between its own minimum demand \( \(d_L\) \) and maximum demand \( \(d_H\) \).

First consider how an increase in the risk aversion rate of both firms affects their expected marginal utilities. Firm A’s initial equilibrium order is very large, hence its own risk aversion effect is significant, however its strong profitability (and/or low risk aversion rate) cancels out the effect of risk aversion increment. Hence the effect of own risk aversion for firm A is not going to be materialized. On the other hand firm B will realize the effect of the risk aversion increment.

Next consider the effect of spillover demand. In this case firm B has no spillover demand, since firm A’s order quantity is large \( \(Q^A > d_H\) \). On the other hand, the effect of demand spillover is influential for firm A. Given that risk aversion impact is not an issue for firm A hence even slight reduction of order quantity by firm B, results in that firm A increases its order quantity. Aggregating these two effects, firm B reduces its order quantity due to risk aversion increment while firm A increases its order quantity given the reduction of firm B’s order quantity.

**Theorem 4** For concreteness, let \( i = A \) and \( j = B \). Suppose the primary demands of the two firms are perfectly negatively correlated. As both firms become more risk averse, Firm A increases its order quantity if and only if the initial equilibrium is one of the following types:

1. Firm A orders minimum demand plus spillover and firm B orders between its minimum demand plus spill over and maximum primary demand

\[
Q^A = d_L + b^A(d_H - Q^B) \quad \text{and} \quad Q^B = d_L + b^B(d_H - Q^A) + \frac{\ln \left( \frac{C_B}{C_A} \left( \frac{1 - qL}{qL} \right) \right)}{R^B} \leq d_H.
\]

This is the unique equilibrium if the following conditions hold

\[
\frac{C^A}{C^B} \leq \frac{qL}{1-qL} = t_{LL}^A,
\]

\[
t_{LL}^B = \frac{qL}{1-qL} < \frac{C^B}{C^A} \leq t_{LL}^B e^{R^B(d_H^B(Q^A) - d_H^B(Q^A))},
\]

where

\[
t_{LL}^B e^{R^B(l_H^B(Q^A) - l_H^B(Q^A))} = \frac{qL}{1-qL} \exp \left( - R^B \left( \frac{(d_H - d_L) (1 - b^B) - b^A b^B}{1 - b^A b^B} \ln \left( \frac{C_B}{C_A} \left( \frac{1 - qL}{qL} \right) \right) \right) \right).
\]

Note that \( \frac{C^B}{C^A} < t_{LL}^B e^{R^B(d_H^B(Q^A) - d_H^B(Q^A))} \) is equivalent to

\[
\frac{C^A}{C^B} < \frac{qL}{1-qL} e^{(1-b^B)(d_H - d_L)}.
\]
2. Both firms’ order quantities are between their minimum demand plus spillover and their maximum demand

\[ Q^A = d_L + b^A(d_H - Q^B) + \frac{\ln \left( \frac{C^A_o}{C^A} \left( \frac{1-qL}{qL} \right) \right)}{R^A} \quad \text{and} \quad Q^B = d_L + b^B(d_H - Q^A) + \frac{\ln \left( \frac{C^B_o}{C^B} \left( \frac{1-qL}{qL} \right) \right)}{R^B}. \]

This is the unique equilibrium if the following conditions hold

\[ t^{iLL} = \frac{q}{1-q} < \frac{C^i_o}{C^i} \leq t^{iLL}e^{\Gamma(d_{iH}^L(Q^i) - d_{iH}^L(Q^j))} \text{ for } i \neq j, i, j = \{A, B\}. \]

In addition it must be that \( Q^A < d_H \), and the following condition must also hold:

\[ \frac{C^B}{C^A} > \frac{qL}{1-qL} \left[ \frac{C^A}{C^A} \left( \frac{1-qL}{qL} \right) \right]^{\frac{R^B}{R^A}}, \quad (34) \]

where

\[ t^{iLL}e^{\Gamma(d_{iH}^L(Q^i) - d_{iH}^L(Q^j))} = \frac{qL}{1-q} e^{-d_H - d_{dL} - b^d dL + b^d dL + b^d dL + b^d dL} \frac{\ln \left( \frac{C^i_o}{C^i} \left( \frac{1-qL}{qL} \right) \right)}{R^i}. \]

Note that \( \frac{C^i_o}{C^i} \leq t^{iLL}e^{\Gamma(d_{iH}^L(Q^i) - d_{iH}^L(Q^j))} \) is equivalent to

\[ \frac{C^i_o}{C^i} \leq \frac{qL}{1-qL} e^{d_H - d_{dL} - b^d dL + b^d dL + b^d dL + b^d dL} \frac{\ln \left( \frac{C^i_o}{C^i} \left( \frac{1-qL}{qL} \right) \right)}{R^i} \]

Similar statement, as above, is valid for firm B.

When the demand is perfectly negatively correlated, firm A only experiences low demand plus potential spillover \( (d_L + b^A[d_H - Q^B]^+) \) or high demand \( (d_H) \). Therefore firm A’s order quantity can be the low demand plus spillover, between low demand plus spillover and high demand, or High demand.

As long as firm A’s competitor (Firm B) orders less than high demand \( (d_H) \), firm A’s order quantity depends on firm B’s order quantity. In two cases we observe such behavior.

In other cases, since firm A’s order quantity is independent of firm B’s order quantity, hence as both firms become more risk averse, firm A does not increase its order quantity. In these cases the only factor that has an impact on firm A’s decision is the own risk aversion increment.

However if firm A’s order quantity depends on its competitor’s order quantity, then firm A may increase its order quantity as both firms become risk averse. We will investigate these two cases in detail considering different situations. We will argue that when firm A increases its order quantity, the impact of spillover demand exceeds that of its own risk-aversion increment; as result, firm A
increases its order quantity. Note that these two cases are comparable to Cases 2 and 3 of Theorem 2.

Observe that Cases 1, 4, 5, and 6 of Theorem 2 are irrelevant under perfect negative correlation, since both firms order at least the minimum demand plus spillover, and at most the maximum demand. As a result, these equilibrium cases cannot occur under perfect negative correlation.

Now we study Theorem 3 and Theorem 4 in more detail. In particular, we investigate three important circumstances: 1. Different understocking to overstocking ratios ($C_u^i/C_o^i$); 2. different risk aversion rates ($R_i$); and 3. different spillover rates ($b_i$).

1. Different understocking to overstocking ratios ($C_u^i/C_o^i$). In this case both firms are identical except that they have different profitability ratio ($C_u^i/C_o^i$). Note that the order quantity of the more profitable firm is more than or equal to the less profitable firm’s order quantity at the equilibrium.

In case of perfect positive correlation, it is straightforward to show that it is the more profitable firm that increases its order quantity if any of them increases its order quantity as both firms become more risk averse. As mentioned above, the only case in which a firm may increase its order quantity as both firms become more risk averse, is the case where the firm orders the maximum demand plus spillover ($d_H + b^i(d_H - Q^H)$) while its competitor orders less than maximum demand ($d_H$). To be able to order the maximum demand plus spillover, since both firms are identical except that they have different profitability ratio, the firm must be more profitable while its competitor’s profitability should be in intermediate range to avoid ordering its own maximum demand ($d_H$). As mentioned before this is very similar to Case 5 of Theorem 2. Hence the discussion of Case 5 in Section 4.1 applies here as well.

In case of perfect negative correlation, it is the less profitable firm that increases its order quantity if either one increases its order quantity, as both firms become more risk averse.

Under the equilibrium with initial risk aversion levels, firm A, less profitable firm, orders less and has lower spillover demand than firm B.

First consider how an increase in the risk aversion rates of both firms affects their expected marginal utilities. Since firm A’s initial equilibrium order is relatively lower, its own risk aversion effect is relatively smaller than firm B. Firm B gets hit harder by its own risk aversion effect since its risk exposure is larger due to its larger initial order.

Next consider the effect of spillover demand. Since firm A’s spillover demand is relatively smaller at the initial equilibrium, its expected marginal utility is more sensitive to a change in its competitor’s order. In particular, a given reduction in firm B’s order significantly reduces firm A’s expected marginal utility loss due to overstocking. By contrast, since firm B’s spillover demand is
relatively larger at the initial equilibrium, its expected marginal utility is less sensitive to a change in the order of firm A. Therefore, the magnitude of the spillover demand effect per unit change in the competitor’s order is significantly larger for firm A than for firm B. A relatively small reduction in firm B’s order translates into a spillover demand effect for firm A which is large enough to offset firm A’s relatively small negative own risk aversion effect. Firm B, however, requires a relatively more significant reduction in firm A’s order for firm B’s spillover demand effect to offset its own risk aversion effect, because firm B has both a lower marginal sensitivity to more spillover demand and a more negative own risk aversion effect. Therefore, the net effect is for firm A to increase and for firm B to decrease its equilibrium order.

In particular, in Part 2 of Theorem 4, the inequality (34) simplifies as follows:

$$\frac{C_u^B}{C_o^B} > \frac{q_L}{1 - q_L} \left[ \frac{C_u^A}{C_o^A} \left( \frac{1 - q_L}{q_L} \right) \right]^{\frac{1}{R^A}}.$$ 

Therefore, to have above inequality, firm A should be less profitable compare to firm B to increases its order quantity as both firm become more risk averse.

As noted above, Parts 1 and 2 of Theorem 4 are very similar to Cases 2 and 3 of Theorem 2 respectively. Hence the discussion of Cases 2 and 3 in Section 4.1 applies here.

2. Different risk aversion rate ($R^i$): In this case both firms are identical except that they have different risk aversion rate ($R^i$). In case of perfect positive correlation, it is straight forward to see that it is less risk averse firm that increases its order quantity if either one increases its order quantity as both firms become more risk averse. Similar argument to that of different understocking to overstocking ratio ($\frac{C_o^B}{C_o^A}$) case will apply here as well, except this time for firm A to order maximum demand plus spillover, it should have initially sufficient low risk aversion rate and its competitor should not have very low risk aversion, to avoid ordering the high demand ($d_H$).

In case of perfect negative correlation, Part 1 of Theorem 4 is not applicable under this assumption, since both firms have the same profitability ratio. Therefore, either both firms order the minimum demand plus spillover or neither. However conditions of Part 2 of Theorem 4 imply that it is more risk averse firm that increases its order quantity if any of them increases its order quantity as both firms become more risk averse. In particular, inequality (34) in Theorem 4 simplifies as follows:

$$\frac{C_u}{C_o} \left( \frac{1 - q_L}{q_L} \right) > \left[ \frac{C_u}{C_o} \left( \frac{1 - q_L}{q_L} \right) \right]^{\frac{R^B}{q(R^3)^2}}.$$ 

Therefore, firm A must be initially more risk averse than firm B, if it is to increases its order quantity as both firms become more risk averse. A similar argument as that for the case of different
understocking to overstocking ratios \( \left( \frac{C_4}{C_5} \right) \) applies here as well. Under the equilibrium with initial risk aversion levels, firm A therefore orders less and has lower spillover demand than firm B.

Similar to different profitability ratio, first consider how an increase in the risk aversion rate \( R \) of both firms affects their expected marginal utilities. Since firm A’s initial equilibrium order is relatively lower, its own risk aversion effect is relatively smaller than firm B. In other words, firm B gets hit harder by its own risk aversion effect since its risk exposure is larger due to its larger initial order.

Similarly we consider the effect of spillover demand. Since firm A’s spillover demand is relatively smaller at the initial equilibrium, its expected marginal utility is more sensitive to a change in its competitor’s order. In particular, a given reduction in firm B’s order significantly reduces firm A’s expected marginal utility loss due to overstocking. By contrast, since firm B’s spillover demand is relatively larger at the initial equilibrium, its expected marginal utility is less sensitive to a change in the order of firm A. Therefore, the magnitude of the spillover demand effect per unit change in the competitor’s order is significantly larger for firm A than for firm B. A relatively small reduction in firm B’s order translates into a spillover demand effect for firm A which is large enough to offset firm A’s relatively small negative own risk aversion effect. Firm B, however, requires a relatively more significant reduction in firm A’s order for firm B’s spillover demand effect to offset its own risk aversion effect, because firm B has both a lower marginal sensitivity to more spillover demand and a more negative own risk aversion effect. Therefore, aggregate impact is for firm A to increase and for firm B to decrease its equilibrium order.

3. Different spillover rate \( (b^i) \): In this case both firms are identical except that they have different spillover rate \( (b^i) \). In such circumstances, firm A will not increase its order quantity given that the stated conditions in Theorem 3 and 4 never hold.

As discussed in Section 4.3, varying the spillover rate \( (b^i) \) has a less significant effect than changing profitability ratio or initial risk aversion ratio. In the perfect correlation case, we observe the extreme situation. Neither firm increases its order quantity as both firms become more risk averse when both firms are identical except for their spillover rates. As discussed above, firm A increases its order quantity if it has significantly higher order quantity in the perfect positive correlation case or lower order quantity in the perfect negative correlation case at the equilibrium compared to its competitor. However since spillover rate has a secondary effect compared to profitability ratio and initial risk aversion rate, when both firms are identical except that they have different spillover rate, the difference between two firms order quantity is not significantly high, hence neither of the firm increase its order quantity as both become more risk averse.
Next we present the effect of perfect correlation on impact of risk aversion graphically. Figure 7 compares three cases of primary demand correlation: 1) Perfect positive correlation; 2) independent demands; and 3) perfect negative correlation. Note that, under perfect positive primary demand correlation, if any firm increases its order quantity as both firms become more risk averse, it must be the more profitable one. When demands are perfectly negatively correlated, then it is the less profitable firm that may increase its order quantity as both firms become more risk averse. This observation is somewhat intuitive: When the demands of the two firms are perfectly positively correlated, this means that if the firm’s competitor faces high (low) demand, firm’s realized demand will be high (low) as well. Furthermore, we know that the firm will only get spillover demand if its competitor faces high demand. But if the competitor faces high demand, then the firm also faces high demand. So there is only one way for the firm to benefit from spillover demand, and that is when the firm’s initial order exceeds its own high demand ($d_H$). As discussed above, the firm should be highly profitable to be willing to order more than its own high demand ($d_H$). Therefore the only situation that "spillover demand" effect may dominate the "increasing own risk aversion" effect and firm increase its order quantity is when the firm is highly profitable or has very low risk aversion quantity is when the firm is highly profitable or has very low risk aversion.

![Diagram](image1.png) (a) Perfect Positive Correlation.  
![Diagram](image2.png) (b) Independent Demand (Base Case).  
![Diagram](image3.png) (c) Perfect Negative Correlation.

Figure 7: Impact of Risk Aversion depending on correlation structure in understocking to overstocking space $\frac{C_F}{C_D}$. Parameters: $K^d = 0.5$, $R^d = 0.3$, $b^d = 0.5$, $q = 0.4$, $d_L = 5$, $d_H = 10$.

On the other extreme end, if the firms’ demands are perfectly negatively correlated, then when the firm’s competitor faces high (low) demand, the firm would face low (high) demand. Hence the maximum possible demand a firm may realize in this situation is its own maximum demand ($d_H$); as a result, the firm is never going to order more than $d_H$. Therefore, if the firm is highly profitable, it will at most order its own maximum demand, so the effect of spillover demand does not arise. Therefore, the highly profitable firm will not increase its order quantity as both firms become more risk averse.
Furthermore, note that the minimum possible realized demand is different due to negative correlation, compared to the independent demand case. The firm’s minimum realized demand is at least its own low demand plus spillover \((d_L + b_i(d_H - Q^i))\). From the independent demand case, we know that when the firm orders between its own low demand plus spillover and its high demand, it is the less profitable firm that may increase its order quantity as both firms become more risk averse. It turns out we have a similar case here, i.e., it is only the less profitable firm that may increase its order quantity as both firm becomes more risk averse.

7 Summary

We have studied the impact of the risk aversion on firms’ order quantities. We have characterized the best response function and proved the existence and uniqueness of Nash equilibrium under general demand distribution and utility function. We have then shown that under competition, one of the firms may increase its order quantity as both firms become more risk averse. To gain specific insights on how risk aversion affects the equilibrium depending on the model parameters, we chose two point demand distribution and CARA utility function. Table 1 summarizes our results. It shows which firm may increase its order quantity as both firms become more risk averse.

<table>
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<th>Perfect Negative Correlation</th>
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<td>Both Higher or Lower (\frac{C_i}{C_o})</td>
<td>Higher (\frac{C_i}{C_o})</td>
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<tr>
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<td>Initially Higher (R)</td>
<td>Initially Both Higher or Lower (R)</td>
<td>Initially Lower (R)</td>
</tr>
<tr>
<td>Different (b)</td>
<td>Neither</td>
<td>Both Lower or Higher (b)</td>
<td>Neither</td>
</tr>
</tbody>
</table>

Table 1: Summary of Results
References


8 Appendix: Proofs

Proof of Lemma 1. It is immediate by definition of $U_k(Q)$ that $U(Q) = U_k(Q)$ on $[d_k, d_{k+1}]$.

Part 1. That $U$ is continuous in $Q$ follows by continuity of $\Pi(Q,d)$ and continuity of $u$. The first derivative of $U_k$ satisfies

$$U'_k(Q) = -C_0 \sum_{i=1}^k q_i u'((C_u + C_o)d_i - C_oQ) + C_u \left(1 - \sum_{i=1}^k q_i \right) u'(C_uQ), \quad (35)$$

where $U'_0(Q) > 0 > U'_N(Q)$ for all $Q$ since $u' > 0$. That $U''_k < 0$ follows since $u'' < 0$. Noting that $U'_{k-1}(Q) > U'_k(Q)$ for all $k$ and $Q$, it follows that $U$ is strictly concave as claimed, with $U''(Q) = U''_0(Q) < 0$ for $Q \in (d_k, d_{k+1})$ and $U'_-(d_k) = U'_{k-1}(d_k) > U'_k(d_k) = U'_+(d_k)$ for $Q = d_k$.

Part 2. Since $U$ is strictly concave it has an unique maximum $Q^*$. It satisfies $U'_{k-1}(d_k) = 0 \geq U'_k(d_k)$ if $Q^* = d_k$, and $U'(Q^*) = U'_0(Q^*) = 0$ if $Q^* \in (d_k, d_{k+1})$ for some $k$. Therefore it is the largest order quantity at which the left derivative $U'_-$ is non-negative: $Q^* = \max \{Q \geq 0 : U'_-(Q) \geq 0\}$, where $Q^* \in [d_1, d_N]$ since $U'_-(d_1) = U'_0(d_1) > 0$ and $U'_-(Q) = U'_N(Q) < 0$ for $Q > d_N$. Calculating $U'_-(Q)$ and rearranging terms yields the condition (5).

\[\square\]

Proof of Lemma 2. Recall the expected utility function (2)

$$U^i(Q^i|Q^j) = \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} q_{xy} u^i(\Pi^i(Q^i, d_{xy}(Q^j))). \quad (36)$$

The payoff function (1) satisfies

$$\Pi^i(Q^i, d) = ((C^i_u + C^i_o)d - C^i_oQ^i) 1\{d < Q^i\} + C^i_uQ^i 1\{d \geq Q^i\}. \quad (37)$$

Continuity and piecewise differentiability of $U^i(Q^i|Q^j)$ in $Q^j$ follow since $\Pi^i$ and $u^i$ have these same properties. Continuity and piecewise differentiability of $Q^i(Q^j)$ in $Q^j$ follow since $U^i(Q^i|Q^j)$ is concave in $Q^i$ for every $Q^j$, and $U^i(Q^i|Q^j)$ is continuous and piecewise differentiable in $(Q^i, Q^j)$.

Part 1. Since $u'' > 0$, $\Pi^i$ is non-decreasing in $d$, and $d_{xy}(Q^j)$ are non-increasing in $Q^j$, $U^i(Q^i|Q^j)$ is constant in $Q^j$ iff $\Pi^i(Q^j, d_{xy}(Q^j))$ is constant in $Q^j$, and strictly decreasing otherwise. For fixed $Q^i$ the payoff $\Pi^i(Q^i, d)$ is strictly increasing in $d < Q^i$ and constant in $d \geq Q^i$.

$d_{xy}(Q^j)$ has the following form: $d_{xy}(Q^j) = d_x^i + b^i(d_y - Q^j)^+$. Therefore $u^i(\Pi^i(Q^i, d_{xy}(Q^j)))$ could be constant or decreasing in $Q^i$. Consequently $U^i(Q^i|Q^j)$ is constant or decreasing in $Q^j$.

Part 2. The claim is: $0 \geq \frac{\partial Q^i}{\partial Q^j} \geq -b$. Similar to equation (5), define

$$f(Q^i, Q^j) = \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d_{xy}(Q^j) < Q^i\} \cdot q_{x,y} \cdot \left(\frac{u'(((C_u + C_o)d_{xy}(Q^j) - C_oQ^j))}{u'(C_uQ^j)}C_o + C_u\right) - C_u, \quad (38)$$
Based on Lemma 1 the optimal order quantity for firm $i$ for given $Q^j$ is unique and given by

$$Q^i(Q^j) = \max\{Q^j \geq 0 : f(Q^i, Q^j) \leq 0\}. \quad (39)$$

Notice that there are two possibilities: either $Q^i(Q^j)$ equals one of the total demand realizations, or $Q^i(Q^j)$ is between two total demand realizations. To prove the claim we consider two cases: Case A. when $Q^i(Q^j)$ is equal to a total demand realization, and Case B when $Q^i(Q^j)$ is between two total demand realizations.

Before we prove the claim for Case A, we present four key properties of the function $f(Q^i, Q^j)$.

**Property 1.** For fixed $Q^i$, $f(Q^i, Q^j)$ is non-decreasing in $Q^j$ with possible jumps in $f(Q^i, Q^j)$'s value.

By (38), $f(Q^i, Q^j)$ is equal to the sum of the following summands minus the constant $C_u$:

$$1\{d_{xy}^i(Q^j) < Q^i\} \cdot q_{xy} \cdot \left(\frac{u' \left((C_u + C_o)d_{xy}^i(Q^j) - C_oQ^j\right)}{u'(C_uQ^i)}C_o + C_u\right). \quad (40)$$

As $Q^j$ increases, we observe two effects: (a) potential change in the value of the indicator function in (40) (b) potential change in value of

$$u' \left((C_u + C_o)d_{xy}^i(Q^j) - C_oQ^j\right).$$

As $Q^j$ increases, $d_{xy}^i(Q^j)$ remains constant or decreases. If the condition of the indicator function is already satisfied, it will remain satisfied since $d_{xy}^i(Q^j)$ remains constant or decreases. However if the condition of indicator function is not already satisfied, it may become satisfied. For fixed $Q^i$, if $d_{xy}^i(Q^j)$ decreases the condition in the indicator function of expression (40) might become satisfied (effect (a)). As a result the number of summands of function $f(Q^i, Q^j)$ might increase. Notice that as the indicator function becomes 1 for a total demand realization, the value of $f(Q^i, Q^j)$ jumps up. Also note that

$$u' \left((C_u + C_o)d_{xy}^i(Q^j) - C_oQ^j\right)$$

in expression (40) is a non-decreasing function of $Q^j$ for fixed $Q^i$ (effect (b)). Therefore, both effects (a) and (b) show that $f(Q^i, Q^j)$ is a non-decreasing function of $Q^j$.

**Property 2.** For fixed $Q^j$, $f(Q^i, Q^j)$ is increasing in $Q^i$.

As $Q^i$ increases, we observe two effects: (a) potential change in the value of the indicator function in (40) (b) potential change in value of

$$\frac{u' \left((C_u + C_o)d_{xy}^i(Q^j) - C_oQ^j\right)}{u'(C_uQ^i)}.$$
Again consider expression (40). As \( Q^i \) increases and \( Q^j \) remains fixed, the condition in the indicator function might be satisfied for more total demand realizations (effect (a)), since \( Q^i \) increases and the total demand realizations remain the same (\( Q^j \) remains fixed). Also note that

\[
\frac{u' \left( (C_u + C_o)d_{xy}^j(Q^j) - C_oQ^j \right)}{u' (C_uQ^j)}
\]

in expression (40) is an increasing function of \( Q^i \) for fixed \( Q^j \). As \( Q^i \) increases, the numerator increases and denominator decreases. Therefore, in light of effects (a) and (b) we can conclude \( f(Q^i, Q^j) \) is increasing in \( Q^i \) for fixed \( Q^j \).

**Property 3.** Consider the case that the best response is equal to a total demand realization that involves spillover demand (i.e. \( Q^i(Q^j) = d_{sp}^i(Q^j) = d_s^i + b^i(d_s^i - Q^j) \) where \( d_s^i > Q^j \)) and remains equal to this total demand realization as \( Q^j \) changes, we show that \( \frac{df(Q^i, Q^j)}{dQ^j} < 0 \).

In this case as \( Q^j \) increases infinitesimally the number of summands in (38) remains the same. The condition of indicator function in (38) remain unsatisfied for those demand points that are larger than or equal to \( d_{sp}^i(Q^j) \). Although those demand points that are larger than or equal to \( d_{sp}^i(Q^j) \) may decrease as \( Q^j \) increases, \( Q^i(Q^j) \) also decreases with the same rate so the ranking of those total demand realizations with respect to \( d_{sp}^i(Q^j) \) remain the same. Those demand points that are less than \( d_{sp}^i(Q^j) \), also remain less than \( d_{sp}^i(Q^j) \) as infinitesimal increase in \( Q^j \) keeps the ranking of the total demand realizations that are smaller than \( d_{sp}^i(Q^j) \) with respect to \( d_{sp}^i(Q^j) \) intact. Therefore the condition of the indicator function in (38) for the total demand realizations smaller than \( d_{sp}^i(Q^j) \) remain satisfied. As a result, to show Property 3, we only need to focus on changes in the following ratio for the total demand realizations \( d_{xy}^i(Q^j) \)s such that \( d_{xy}^i(Q^j) < d_{sp}^i(Q^j) \):

\[
h(Q^i, Q^j) = \frac{u' \left( (C_u + C_o)d_{xy}^j(Q^j) - C_oQ^j \right)}{u' (C_uQ^j)}
\]

\[
= \frac{u' \left( (C_u + C_o)d_{xy}^j(Q^j) - C_o \left( d_s^i + b^i(d_s^i - Q^j) \right) \right)}{u' \left( C_u \left( d_s^i + b^i(d_s^i - Q^j) \right) \right)}
\]

Taking the derivative of \( h(Q^i, Q^j) \) with respect to \( Q^j \), we have:

\[
\frac{dh(Q^i, Q^j)}{dQ^j} = \frac{u'' \left( (C_u + C_o)d_{xy}^j(Q^j) - C_o \left( d_s^i + b^i(d_s^i - Q^j) \right) \right) \left( C_o b^i - (C_u + C_o)b^i \cdot 1 \{d_y^j > Q^j \} \right)}{u' \left( C_u \left( d_s^i + b^i(d_s^i - Q^j) \right) \right)}
\]

\[
- \frac{-u' \left( (C_u + C_o)d_{xy}^j(Q^j) - C_o \left( d_s^i + b^i(d_s^i - Q^j) \right) \right)}{u' \left( C_u \left( d_s^i + b^i(d_s^i - Q^j) \right) \right)} \left( -C_u b^i \right)
\]

\[
\left( u' \left( C_u \left( d_s^i + b^i(d_s^i - Q^j) \right) \right) \right)^2
\]
replacing \( Q^i = d^i_s + b^i(d^i_p - Q^i) \), we have:

\[
\frac{dh(Q^i, Q^i)}{dQ^i} = \frac{u''((C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i)}{u'(C_oQ^i)} \times \left[ (C_o b^i - (C_u + C_o)b^i \cdot 1\{d^i_y > Q^j\}) - (-C_u b^i) \frac{u'((C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i)}{u''((C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i)} \frac{u'(C_oQ^i)}{u'(C_oQ^i)} \right]
\]

Define

\[
g(x) = \frac{u''(x)}{u'(x)}
\]

Substituting \( g(x) \) in \( \frac{dh(Q^i, Q^i)}{dQ^i} \), we have:

\[
\frac{dh(Q^i, Q^i)}{dQ^i} = \frac{u''((C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i)}{u'(C_oQ^i)} \times b^i \left[ (C_o + C_u) \frac{g(C_oQ^i)}{g((C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i)} \right) - (C_u + C_o) \cdot 1\{d^i_y > Q^j\}] \]

Since (1) \( g'(x) \geq 0 \) (based on our assumption) and (2)

\( C_u Q^i \geq (C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i \)

because \( Q^i(Q^i) = d^i_{sp}(Q^i) \geq d^i_{xy}(Q^i) \) as mentioned before, we have

\[
g(C_oQ^i) > 1.
\]

Therefore \( \frac{dh(Q^i, Q^i)}{dQ^i} < 0 \), since inside the bracket in Equation (41) is positive and

\[
\frac{u''((C_u + C_o)d^i_{xy}(Q^i) - C_oQ^i)}{u'(C_oQ^i)} < 0.
\]

**Property 4.** \( f(Q^i, Q^j) \) is continuous in \( Q^j \) except for the points that the indicator function in (38) become satisfied for a total demand realization. Consider one of the summands in (38):

\[
1\{d^i_{xy}(Q^j) < Q^j\} \cdot q_{x,y} \cdot \left( u'((C_u + C_o)d^i_{xy}(Q^j) - C_oQ^i) \right) \frac{(C_o + C_u)}{u'(C_oQ^i)}
\]

Since \( u'((C_u + C_o)d^i_{xy}(Q^j) - C_oQ^i) \) is a continuous function in \( Q^j \), if the values of indicator function does not change, the above expression is continuous in \( Q^j \), otherwise we have a discontinuity when the value of the indicator function changes. Based on this argument if the value of the indicator function does not change in (38), \( f(Q^i, Q^j) \) is continuous in \( Q^j \), otherwise we have discontinuity when the value of the indicator function changes.

**Case A.** We now prove the claim for the case where the best response is equal to a total demand realization i.e. \( f(Q^i, Q^j) \leq 0 \) for \( Q^j \leq d^i_{sp}(Q^j) \) and \( f(Q^i, Q^j) > 0 \) for \( Q^j > d^i_{sp}(Q^j) \). To prove the claim, we consider two cases: A.1 when the best response is equal to a total demand realization that involves spillover demand, i.e. \( Q^i(Q^j) = d^i_{sp}(Q^j) = d^i_s + b^i(d^i_p - Q^j) \) where \( d^i_p > Q^j \), and A.2
when the best response is only equal to a total demand realization that does not involve spillover demand, i.e. \( Q^i(Q^i) = d^i_{sp}(Q^i) = d^i_s \).

**Case A.1.** Consider the case where the best response is equal to a total demand realization that involves spillover demand (i.e. \( Q^i(Q^i) = d^i_{sp}(Q^i) = d^i_b + b^i(d^i_s - Q^i) \), where \( d^i_s > Q^i \) and remains on this demand point as the competitors order quantity changes. As shown in Property 3, \( f(Q^i, Q^j) \) is a decreasing function of \( Q^j \) in this case. Hence for small \( \varepsilon > 0 \), \( f(Q^i, Q^i + \varepsilon) < 0 \). Therefore firm \( i \)'s best response should be larger than or equal to \( d^i_{sp}(Q^i + \varepsilon) \) when the competitor orders \( Q^i + \varepsilon \). If \( Q^i(Q^i + \varepsilon) \) is equal to \( d^i_{sp}(Q^i + \varepsilon) = d^i_s + b^i(d^i_b - Q^i - \varepsilon) \), we have \( \frac{\partial Q^i}{\partial Q^j} = -b \), otherwise \( Q^i(Q^i + \varepsilon) \) must be larger than \( d^i_{sp}(Q^i + \varepsilon) \). This means that \( Q^i(Q^i + \varepsilon) \) is between two demand points which shows that \( -b < \frac{\partial Q^i}{\partial Q^j} < 0 \).

**Case A.2.** Consider the case where firm \( i \)'s best response function is only equal to a total demand realization that does not involve spillover demand (i.e. \( Q^i(Q^i) = d^i_{sp}(Q^i) = d^i_s \)) and remains on this demand point as the competitor's order quantity changes. We break down this case to two sub cases: Case A.2.a. when \( f(Q^i, Q^j) < 0 \), and Case A.2.b. when \( f(Q^i, Q^j) = 0 \).

**Case A.2.a.** If \( f(Q^i, Q^j) < 0 \) for \( Q^i(Q^i) = d^i_{sp}(Q^i) = d^i_s \) then we show that Firm \( i \)'s best response stays on this demand point for small changes in its rival's order quantity, i.e., \( Q^i(Q^i + \varepsilon) = d^i_{sp}(Q^i + \varepsilon) \) for small \( \varepsilon > 0 \). If \( d^i_{sp}(Q^i) \) is constant (i.e. \( d^i_{sp}(Q^i) = d^i_s \)), small increases in \( Q^i \), results in either very small increase or no change in \( f(Q^i, Q^j) \) (as shown in Property 1). However since the change in \( f(Q^i, Q^j) \) is infinitesimal and \( f(Q^i, Q^j) \) is continuous in \( Q^j \) based on Property 4 (since the indicator functions in \( f(Q^i, Q^j) \) remain the same as \( Q^i(Q^i) \) and \( d^i_{sp}(Q^i) \) are constant and all other demand points remain the same or decrease infinitesimally as \( Q^j \) increases infinitesimally), \( f(Q^i, Q^i + \varepsilon) \) remains negative. On the other hand based on (39) we have \( f(Q^i, Q^i + \varepsilon) > 0 \) for \( Q^i > d^i_{sp}(Q^i + \varepsilon) \). Note that the following holds: \( f(Q^i, Q^i + \varepsilon) > 0 \) for \( Q^i > d^i_{sp}(Q^i + \varepsilon) = d^i_s \) since as \( Q^i \) increases \( d^i_{sp}(Q^i) \) remains the same and based on Property 1, \( f(Q^i, Q^j) \) is an increasing function of \( Q^j \). Therefore, we can conclude that \( f(Q^i, Q^i + \varepsilon) > f(Q^i, Q^i) > 0 \) for \( Q^i > d^i_{sp}(Q^i + \varepsilon) = d^i_s \) which shows that the best response function remains at this demand point and as a result \( \frac{\partial Q^i}{\partial Q^j} = 0 \).

**Case A.2.b.** If \( f(Q^i, Q^j) = 0 \) for \( Q^i = d^i_{sp}(Q^i) = d^i_s \). We show that, as \( Q^j \) increases, \( Q^i \) may decrease, and does not necessarily remain on demand point \( d^i_{sp}(Q^i) \). In the case where \( d^i_{sp}(Q^i) \) is constant in \( Q^i \) (i.e. \( d^i_{sp}(Q^i) = d^i_s \)), as \( Q^j \) increases, \( f(Q^i, Q^j) \) may increase or remain the same (shown in Property 1). If \( f(Q^i, Q^j) \) remains the same, the best response remains on this demand point so the claim holds since \( \frac{\partial Q^i}{\partial Q^j} = 0 \). However, if \( f(Q^i, Q^j) \) increases, then we will have \( f(Q^i, Q^i + \varepsilon) > 0 \) at \( Q^i(Q^i) = d^i_{sp}(Q^i) \). As a result we can conclude that firm \( i \)'s best response can not be equal to \( d^i_{sp}(Q^i) \). Based on Property 2, we know that \( f(Q^i, Q^j) \) is an increasing
function of $Q^i$ for given $Q^j$, hence the new best response must be less than $d_{sp}^i(Q^j)$. Therefore this can be considered as a limiting case of the situation in which $Q^i(Q^j)$ is between two total demand realizations, which will be discussed next.

**Case B.** We, now, show $0 \geq \frac{\partial Q_i}{\partial Q^j} \geq -b$ when $Q^i(Q^j)$ is between two total demand realizations. Note that in this case $f(Q^i, Q^j)$ is well defined and differentiable with respect to both $Q^i$ and $Q^j$. Taking the total derivative of $f(Q^i, Q^j)$ with respect to $Q^j$, we have:

$$\frac{df(Q^i, Q^j)}{dQ^j} = \frac{\partial f(Q^i, Q^j)}{\partial Q^j} + \frac{\partial f(Q^i, Q^j)}{\partial Q^i} \frac{\partial Q^i}{\partial Q^j},$$

where

$$\frac{\partial f(Q^i, Q^j)}{\partial Q^j} = \sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d_{xy}^i(Q^j) < Q^i\} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^j)}$$

$$\cdot u''((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^j) (-(C_u + C_o) b \cdot 1\{d_y^j > Q^j\})$$

and

$$\frac{\partial f(Q^i, Q^j)}{\partial Q^i} = -\sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d_{xy}^i(Q^j) < Q^i\} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^j)} \left[ C_u u''((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^j) \right]$$

$$+ C_u u''((C_u Q^j) u'(C_u Q^j) u''((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^j) \right]$$

To find $\frac{\partial Q^i}{\partial Q^j}$, set $\frac{df(Q^i, Q^j)}{dQ^j} = 0$ which results in:

$$\frac{\partial Q^i}{\partial Q^j} = -\frac{\frac{\partial f(Q^i, Q^j)}{\partial Q^j}}{\frac{\partial f(Q^i, Q^j)}{\partial Q^i}}$$

We have $\frac{\partial f(Q^i, Q^j)}{\partial Q^j} \geq 0$ and $\frac{\partial f(Q^i, Q^j)}{\partial Q^i} > 0$ since $u''(x) \leq 0$. Therefore $\frac{\partial Q^i}{\partial Q^j} \leq 0$. Next we show that $\frac{\partial Q^j}{\partial Q^i} \geq -b$:

$$-\frac{\frac{\partial f(Q^i, Q^j)}{\partial Q^j}}{\frac{\partial f(Q^i, Q^j)}{\partial Q^i}} \geq -b \iff \frac{\partial f(Q^i, Q^j)}{\partial Q^j} \leq b \frac{\partial f(Q^i, Q^j)}{\partial Q^i}$$

which is equivalent to showing

$$\frac{\partial f(Q^i, Q^j)}{\partial Q^j} - b \frac{\partial f(Q^i, Q^j)}{\partial Q^i} \leq 0$$

Substituting for $\frac{\partial f(Q^i, Q^j)}{\partial Q^j}$ and $\frac{\partial f(Q^i, Q^j)}{\partial Q^i}$ in the above inequality, we have:

$$\sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d_{xy}^i(Q^j) < Q^i\} \cdot \frac{q_{x,y} C_o}{u'(C_u Q^j)} \left[ C_u u''((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^j) \right] \left[ (-(C_u + C_o) b \cdot 1\{d_y^j > Q^j\}) \right]$$

$$+ b \left( C_o + C_u \left( \frac{u''(C_u Q^j)}{u'(C_u Q^j)} u''((C_u + C_o)d_{xy}^i(Q^j) - C_o Q^j) \right) \right) \leq 0$$
Similar to above, define:

\[ g(x) = \frac{u''(x)}{u'(x)} \]

Substituting \( g(x) \) in inequality (42), we have:

\[
\sum_{x=1}^{N^A} \sum_{y=1}^{N^B} 1\{d_{xy}^i(Q^j) < Q^j\} \cdot \frac{q_{yC}}{u'(C_u + C_o)} u''((C_u + C_o)d_{xy}^i(Q^j) - C_oQ^j) \left[ (-b(C_u + C_o) \cdot 1\{d_{y}^i > Q^j\}) \right]
+ \left[ C_o + C_u \frac{g(C_uQ^i)}{g((C_u + C_o)(d_{xy}^i(Q^j) - C_oQ^i))} \right] \leq 0
\]

The above inequality holds since (a) \( g'(x) \geq 0 \) (based on our assumption with regard to the utility function) and (b)

\[ C_oQ^i \geq (C_u + C_o)d_{xy}^i(Q^j) - C_oQ^i \]

because \( Q^i \geq d_{xy}^i(Q^j) \).

Therefore we have shown the claim for the both possibilities: \( Q^i(Q^j) \) is equal to one of the total demand realizations, or \( Q^i(Q^j) \) is between two total demand realizations.

\[ 0 \geq \frac{\partial Q^j}{\partial Q^i} \geq -b. \]

**Proof of Proposition 1. Equilibrium existence.** It is well-known that there exists at least one Nash equilibrium in submodular two-player games. That \( U^i(Q^i|Q^j) \) is submodular follows because the right and left derivatives of \( u^i(\Pi^i(Q^i,d_{xy}^i(Q^j))) \) with respect to \( Q^j \) are non-increasing in \( Q^j \) for all \( x, y \). The left and right derivatives satisfy, respectively,

\[
u''(\Pi^i(Q^i,d_{xy}^i(Q^j))) \left( -C_o^1 \{ d_{xy}^i(Q^j) < Q^j \} + C_o^1 \{ d_{xy}^i(Q^j) \geq Q^j \} \right) \quad \text{and} \quad u''(\Pi^i(Q^i,d_{xy}^i(Q^j))) \left( -C_o^1 \{ d_{xy}^i(Q^j) < Q^j \} \right).
\]

The claim follows since \( u'' < 0 \) and \( d_{xy}^i(Q^j) \) is non-increasing in \( Q^j \).

**Uniqueness.** From Lemma 2.2 we have \(|f_i^j(Q^j)|, |f_i'^j(Q^j)| \leq b^i \leq 1 \) for \( i \neq j \). Therefore if \( b^i < 1 \) then firm \( i \)'s best response function \( f_i^j(Q^j) \) is a contraction and the equilibrium is unique.

**Proof of Theorem 1.** Fix \( R \) and the equilibrium \( Q^*(R) \).

First note that the best response function of each firm has exactly one piece that satisfies (10). This holds due to two facts that follow from Lemma 2. (i) The best response function of each firm has exactly one or two pieces that satisfy (8). (ii) If two pieces of firm \( i \)'s best response function satisfy (8), i.e., \( g_1^i(\Pi^i(Q^i)\R^i) = g_2^i(\Pi^i(Q^j)\R^j) = Q^*(R) \), then they only agree for
\( Q^{-i} = Q^{-i*}(R) \) and at the initial \( R \) vector, that is \( g^i_1(Q^{-i}, R^i) \neq g^i_2(Q^{-i}, R^i) \) for \( Q^{-i} \neq Q^{-i*}(R) \), and \( g^i_1(Q^{-i*}(R(\delta)), R^i(\delta)) \neq g^i_2(Q^{-i*}(R(\delta)), R^i(\delta)) \) for \( \delta \in (0, \delta) \).

Suppose that \( g^A(Q^{-B}, R^A) \) and \( g^B(Q^{-A}, R^B) \) are pieces of the firm \( A \) and \( B \) best response functions, respectively, which satisfy (8). It is evident from Lemma 2 that \( g^A(Q^{-B}, R^A) \) and \( g^B(Q^{-A}, R^B) \) are continuously differentiable. Let \( \overline{Q}(R(\delta)) = (Q^A(R(\delta)), \overline{Q}^B(R(\delta))) \) denote the order vector that satisfies

\[
\overline{Q}^i(R(\delta)) = g^i(\overline{Q}^{-i}(R(\delta)), R^i(\delta)), \quad i \in \{A, B\}.
\]

(43) for \( \delta \) in a neighborhood of 0, where \( \overline{Q}(R(0)) = Q^*(R(0)) = Q^*(R) \). As we show next, this vector is well defined, but first notice that for \( \delta \neq 0 \) we have \( \overline{Q}(R(\delta)) = Q^*(R(\delta)) \) only if (10) holds.

Taking the total derivative of (43) with respect to \( \delta \) yields

\[
\frac{\partial \overline{Q}^i(R)}{\partial R^A} + \frac{\partial \overline{Q}^i(R)}{\partial R^B} = \frac{\partial g^i_1(Q^{-i}(R), R^i)}{\partial Q^{-i}} \left( \frac{\partial Q^{-i}(R)}{\partial R^A} + \frac{\partial Q^{-i}(R)}{\partial R^B} \right) + \frac{\partial g^i_2(Q^{-i}(R), R^i)}{\partial R^i}, \quad i \in \{A, B\}.
\]

Solving these equations yields

\[
\frac{\partial \overline{Q}^i(R)}{\partial R^A} + \frac{\partial \overline{Q}^i(R)}{\partial R^B} = \frac{\frac{\partial g^i_1(Q^{-i}(R), R^i)}{\partial R^A} \frac{\partial g^i_2(Q^{-i}(R), R^i)}{\partial Q^{-i}} + \frac{\partial g^i_1(Q^{-i}(R), R^i)}{\partial Q^{-i}}}{1 - \frac{\partial g^i_1(Q^{-i}(R), R^i)}{\partial Q^{-i}} \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}}}.
\]

(44)

By Part 2 of Lemma 2, all pieces of the best response functions satisfy

\[
\left| \frac{\partial g^i(Q^{-i}, R^i)}{\partial Q^{-i}} \right| \leq b^i < 1 \text{ and } \left| \frac{\partial g^i(Q^{-i}, R^i)}{\partial Q^{-i}} \right| \leq b^i < 1,
\]

which implies that the partial derivatives in (44) are well defined. Furthermore, the RHS of (44) has the same sign as its numerator, and

\[
\frac{\partial \overline{Q}^i(R)}{\partial R^A} + \frac{\partial \overline{Q}^i(R)}{\partial R^B} > 0 \iff \frac{\partial g^i_1(Q^{-i}(R), R^i)}{\partial R^A} \frac{\partial g^i_2(Q^{-i}(R), R^i)}{\partial Q^{-i}} > \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial R^i}.
\]

To complete the proof, note that for the unique pair of best response function pieces \( g^A(Q^{-B}, R^A) \) and \( g^B(Q^{-A}, R^B) \) that satisfy (10), we have \( \overline{Q}(R(\delta)) = Q^*(R(\delta)) \) for \( \delta \in [0, \delta] \), and

\[
\frac{\partial Q^{ix}_+(R)}{\partial R^A} + \frac{\partial Q^{ix}_-(R)}{\partial R^B} = \frac{\partial \overline{Q}^i(R)}{\partial R^A} + \frac{\partial \overline{Q}^i(R)}{\partial R^B}.
\]

Proof of Corollary 1. To show Part 1 and 2, it is required to show:

\[
\frac{\partial g^i(Q^{-i*}(R), R^i)}{\partial R^i} \leq 0.
\]
Note that based on Pratt Theorem (Pratt (1964)), when a firm become more risk averse we have:

\[ U^i(Q^i|Q^j, R) \geq U^i(Q^i|Q^j, R + \delta), \]

which means the best response for the LHS of the above inequality is at least larger than equal to that of the RHS of the above inequality. This means that

\[ \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial R^i} \leq 0. \]

**Part 1.** If firms are symmetric then

\[ \frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial R^{-i}} = \frac{\partial g^{-i}(Q^{-i}(R), R^i)}{\partial R^i} \leq 0. \]

That these partial derivatives are nonpositive follows from above discussion. If they are zero, then (11) is clearly violated. If the inequality is strict, then (11) is equivalent to

\[ \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}} < -1, \]

which cannot hold by Part 2 of Lemma 2.

**Part 2.** Suppose to the contrary that (11) holds for \( i \) and its rival:

\[ \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}} \frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial Q^{-i}} + \frac{\partial g^{-i}(Q^{-i}(R), R^i)}{\partial R^i} \frac{\partial g^i(Q^{i*}(R), R^{-i})}{\partial Q^i} \geq 0, \]

\[ \frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial Q^i} \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^i} + \frac{\partial g^i(Q^{i*}(R), R^{-i})}{\partial R^{-i}} \geq 0. \]

Multiply the first inequality by \(-\frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial Q^i}\) (which is nonnegative), and add the result to the second to obtain:

\[ \left(1 - \frac{\partial g^i(Q^{-i}(R), R^i)}{\partial Q^{-i}} \frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial Q^i}\right) \frac{\partial g^{-i}(Q^{i*}(R), R^{-i})}{\partial Q^i} > 0. \]

This cannot hold since the bracketed term is positive and its multiplier is nonnegative.

**Proof of Lemma 3** Since \( \lim_{x \to -\infty} u'(x) = 0, \lim_{x \to \infty} u'(-x) = \infty, \) and \( U''_k < 0, \) for \( k \in \{1, 2, ..., N - 1\} \) the maximizer \( Q^*_k \overset{\triangleleft}{=} \arg \max_Q U'_k(Q) \) is well-defined and the unique solution of the first-order condition of \( U'_k(Q) = 0. \) The fact that \( U'_{k-1}(Q) > U'_k(Q) \) for all \( k \) and \( Q, \) and \( U''_k < 0 \) implies that \( Q^*_1 > Q^*_2 > ... > Q^*_N. \) That \( Q^* = \min(Q_m^*, d_{m+1}) \) follows since \( Q^*_k \geq d_k \iff U'_k(d_k) \geq 0. \) Therefore \( U'_m(d_m) \geq 0 > U'_{m+1}(d_{m+1}), \) which implies the claim.

**Proof of Lemma 4.** By Lemma 1 the best response \( f^A(Q^B) \) is unique. The expressions in (18)-(20) are immediate from Lemma 3, by using the appropriate demand distribution to calculate the maximizers \( Q^*_k \) in (15) and the conditions in (16)-(17). For illustration, in Part 1, \( Q^B \geq d_H \)
implies that firm A’s total demand distribution has \( N = 2 \) mass points, \( d_{LL}^A(Q^B) = d_{HH}^A(Q^B) = d \) with probability \( q_L = q_{LL} + q_{LH} \), and \( d_{HH}^A(Q^B) = d_{HH}^A(Q^B) = d \) with probability \( 1 - q_L \). Substitute \( d_1 = d_L \) and \( d_2 = d_H \) with \( q_1 = q_L \) and \( q_2 = q_H \), in (15) to get the maximizer

\[
Q_1^* = \frac{1}{R^1} \ln \left( \frac{C_u^A}{C_o^A q_L \exp \left( -R^1 d_L \right) \left( 1 - q_L \right)} \right) = \frac{1}{R^1} \ln \left( \frac{C_u^A}{C_o^A t_L^A} \right).
\]

This maximizer is the optimal order quantity if and only if \( d_L \leq Q_1^* \leq d_H \), or equivalently, (16) holds for \( k = 1 \). This yields the condition in (18) for \( f^A(Q^B) = Q_1^* \).

In Part 2, \( d_L \leq Q^B < d_H \) implies the total demand distribution (14) with \( N = 4 \) mass points. Substituting \( d_1 = d_{LL}^A \), \( d_2 = d_{LH}^A \), \( d_3 = d_{HH}^A \) and \( d_4 = d_{HH}^A \), along with the corresponding probabilities, into (15)-(17) yields (19)-(20).

**Proof of Proposition 2.** The proof is based on Lemma 4.

We illustrate this by proving Case 1. Suppose that firm B orders \( Q^B < d_H \). We need to show that firm A’s best response \( f^A(Q^B) = d_{LL}^A(Q^B) \) if the conditions \( Q_{LL}^A(Q^B) \leq d_{LL}^A(Q^B) < Q_{LL}^A \) are satisfied. By (19)-(20) of Lemma 4, this holds if

\[
t^A_{LL}(Q^B) \exp(R^1 d^A_{LL}(Q^B)) < \frac{C_u^A}{C_o^A} \leq t^A_{LL}(Q^B) \exp(R^1 d^A_{LL}(Q^B)) \tag{45}
\]

where (noting that \( d^A_{LL} = d_L \))

\[
t^A_{LL} = \frac{q_{LL} \exp(-R^1 d_L)}{1 - q_L},
\]

\[
t^A_{LH}(Q^B) = \frac{q_{LL} \exp(-R^1 d_L) + q_{LH} \exp(-R^1 d^A_{LH}(Q^B))}{1 - q_{LL} - q_{LH}}.
\]

Substituting the thresholds into (45) and rearranging terms yields the conditions \( Q_{LL}^A(Q^B) \leq d_{LL}^A(Q^B) < Q_{LL}^A \). Next, suppose that firm A orders \( Q^A < d_H \). We need to show that firm B’s best response \( f^B(Q^A) = Q_{LL}^B \) if the conditions \( d_L < Q_{LL}^B \leq d_{LL}^B(Q^A) \) are satisfied. By (19)-(20) of Lemma 4, this holds if (noting that \( d_{LL}^B = d_L \))

\[
t^B_{LL} \exp(R^1 d_L) < \frac{C_u^B}{C_o^B} \leq t^B_{LL} \exp(R^1 d^B_{LL}(Q^A)), \tag{45}
\]

where \( t^B_{LL} = \frac{q_{LL} \exp(-R^1 d_L)}{1 - q_L} \), which is equivalent to \( d_L < Q_{LL}^B \leq d_{LL}^B(Q^A) \).

The proofs of the remaining cases follow the same line of argument and are therefore omitted.

**Proof of Corollary 2.** First consider identical firms. Then both firms have exactly the same best response function and equal equilibrium order quantities. It follows from (18)-(20) in Lemma 4 that a firm’s best response function weakly increases in \( C_u^i / C_o^i \) and \( b^i \), and weakly decreases in \( R^i \).
Since the best response functions are weakly decreasing by Lemma 2, it follows that in equilibrium, the firm with higher $C_u^i/C_o^i$, lower $R^i$, or higher $b^i$ orders at least as much as its rival.

**Proof of Theorem 2.** The proof consists of checking for each of the 17 equilibrium cases specified in Proposition 2 whether the necessary and sufficient conditions (10)-(11) of Theorem 1 can be satisfied, and if so under what conditions. Doing so involves two steps.

Step 1. Identify the pair of best response function pieces, $g^A(B^R, A^R), g^B(A^R, B^R)$, that satisfy (10), which we replicate here. Let $R(\delta) = R + \delta e$. For some $\bar{\delta} > 0$ and $\delta \in [0, \bar{\delta})$,

$$Q^i*(R(\delta)) = g^i(Q^{-i*}(R(\delta)), R^i(\delta)) = f^i(Q^{-i*}(R(\delta)), R^i(\delta)), \ i \in \{A, B\}. \quad (46)$$

Step 2. Check whether the functions identified in step 1 satisfy condition (11), which we also replicate here, for $i = A$ and $-i = A$:

$$\frac{\partial g^A(B^*(R), A^R)}{\partial B} \frac{\partial g^B(A^*(R), B^R)}{\partial B} > -\frac{\partial g^A(B^*(R), A^R)}{\partial A}. \quad (47)$$

For reference, we replicate the best response function pieces (21)-(25) of Proposition 2, making their dependence on $R^i$ and $Q^{-i}$ explicit. We suppress these arguments unless they are needed for clarity.

$$d_{LH}^i(Q^{-i}) = d_L + b^i(d_H - Q^{-i}) \text{ and } d_{HH}^i(Q^{-i}) = d_H + b^i(d_H - Q^{-i}), \quad (48)$$

$$Q_M^i(R^i) = \frac{1}{R^i} \ln \left( \frac{C_o^i}{C_o^i qL exp(-R^i d_L)} \right), \quad (49)$$

$$Q_{LL}^i(R^i) = \frac{1}{R^i} \ln \left( \frac{C_o^i}{C_o^i qLL exp(-R^i d_L)} \right), \quad (50)$$

$$Q_{LH}^i(Q^{-i}, R^i) = \frac{1}{R^i} \ln \left( \frac{C_o^i}{C_o^i qLL exp(-R^i d_L) + qLL exp(-R^i d_{LH}^i(Q^{-i}))} \right), \quad (51)$$

$$Q_{HL}^i(Q^{-i}, R^i) = \frac{1}{R^i} \ln \left( \frac{C_o^i}{C_o^i qLL exp(-R^i d_L) + qLL exp(-R^i d_{LH}^i(Q^{-i})) + qHL exp(-R^i d_H)} \right) \quad (52)$$

The partial derivatives of (49)-(52) with respect to $R^i$ are nonnegative and satisfy

$$\frac{\partial Q_{k}^{i}}{\partial R^{i}} = Q_{k}^{i'}(R^{i}) = \frac{d_L - Q_{k}^{i}}{R^{i}} \leq 0 \text{ for } k \in \{M, LH\}, \quad (53)$$

$$\frac{\partial Q_{LH}^{i}}{\partial R^{i}} = \frac{1}{R^{i}} qLL d_L exp(-R^{i} d_L) + qLL d_{LH}^i exp(-R^{i} d_{LH}^i) - \frac{Q_{LH}^{i}}{R^{i}} \leq 0, \quad (54)$$

$$\frac{\partial Q_{HL}^{i}}{\partial R^{i}} = \frac{1}{R^{i}} qLL d_L exp(-R^{i} d_L) + qLL d_{LH}^i exp(-R^{i} d_{LH}^i) + qHL d_H exp(-R^{i} d_H) - \frac{Q_{HL}^{i}}{R^{i}} < 0. \quad (55)$$

The inequality in (54) holds since $Q_{LH}^{i} \geq d_{LH}^i \geq d_L$, the one in (55) since $Q_{HL}^{i} \geq d_H > d_L$. 

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The partial derivatives of (51)-(52) with respect to \( Q^{-i} \) satisfy

\[
\frac{\partial Q^i_{LH}}{\partial Q^{-i}} = -b^i \frac{q_{iL} \exp(-\frac{R}{b} d^i_{LH})}{q_{iL} \exp(-\frac{R}{b} d_L) + q_{LH} \exp(-\frac{R}{b} d^i_{LH})},
\]

\[
\frac{\partial Q^i_{LH}}{\partial Q^{-i}} = -b^i \frac{q_{iL} \exp(-\frac{R}{b} d^i_{LH})}{q_{iL} \exp(-\frac{R}{b} d_L) + q_{LH} \exp(-\frac{R}{b} d^i_{LH}) + q_{HL} \exp(-\frac{R}{b} d_H)}.
\]

**Case 1.** The equilibrium order quantities \( Q^A^* (R) \) and \( Q^B^* (R) \) for case 1 satisfy

\[
Q^A_{LH} (Q^B^* (R), R^A) \leq Q^A^* (R) = d^A_{LH} (Q^B^* (R)) < Q^A_{LL} (R^A) \]

\[
d_L < Q^B^* (R) = Q^B_{LH} (R^B) \leq d^B_{LH} (Q^A^* (R)).
\]

Step 1. The best response function pieces that satisfy (46) are \( g^A = d^A_{LH} \) and \( g^B = Q^B_{LH} \). This is obvious if the inequalities in the equilibrium conditions are strict. If \( Q^A_{LH} (Q^B^* (R), R^A) = Q^A^* (R) = d^A_{LH} (Q^B^* (R)) \) and/or \( Q^B_{LH} (R^B) = d^B_{LH} (Q^A^* (R)) \), this follows since \( Q^B_{LL} (R^B) < 0 \) by (53) and \( \partial Q^A_{LH}/\partial R^A < 0 \) by (54), whereas \( \partial d^i_{LH}/\partial R^i = 0 \) for \( i \in \{A, B\} \).

Step 2. That (47) holds follows since \( g^A = d^A_{LH} \) is constant in \( R^A \) and

\[
\frac{\partial g^B}{\partial R^B} \frac{\partial g^A}{\partial Q^B} = Q^B_{LH} (R^B) \frac{\partial d^A_{LH}}{\partial Q^B} > 0.
\]

**Case 2.** The equilibrium order quantities \( Q^A^* (R) \) and \( Q^B^* (R) \) for case 2 satisfy

\[
Q^A_{LH} (Q^B^* (R), R^A) \leq Q^A^* (R) = d^A_{LH} (Q^B^* (R)) < Q^A_{LL} (R^A),
\]

\[
d^B_{LH} (Q^A^* (R)) < Q^B^* (R) = Q^B_{LH} (Q^A^* (R), R^B) < d_H.
\]

Step 1. The best response function pieces that satisfy (46) are \( g^A = d^A_{LH} \) and \( g^B = Q^B_{LH} \). This follows since \( \partial Q^A_{LH}/\partial R^A < 0 \) by (54), whereas \( \partial d^A_{LH}/\partial R^A = 0 \).

Step 2. That (47) holds follows since \( g^A = d^A_{LH} \) is constant in \( R^A \) and strictly decreasing in \( Q^B \), and from (54) we have

\[
\frac{\partial g^B (Q^A^* (R), R^B)}{\partial R^B} < 0,
\]

because \( Q^B^* (R) < d_H \) implies that \( Q^A^* (R) = d^A_{LH} (Q^B^* (R)) > d_L \).

**Case 3.** The equilibrium order quantities \( Q^A^* (R) \) and \( Q^B^* (R) \) for case 3 satisfy

\[
d^i_{LH} (Q^{-i^*} (R)) < Q^{i^*} (R) = Q^i_{LH} (Q^{-i^*} (R), R^i) \leq d_H, i \in \{A, B\}.
\]

Step 1. We have two cases. If both equilibrium conditions hold with strict inequality, then the best response function pieces that satisfy (46) are \( g^i = Q^i_{LH}, i \in \{A, B\} \). However, if \( Q^{i^*} (R) = Q^i_{LH} (Q^{-i^*} (R), R^i) = d_H \) for \( i = A \) and/or \( i = B \), then the pair \( g^i = Q^i_{LH}, i \in \{A, B\} \) need not satisfy (46), in which case we must have \( g^i = d_H \) for \( i = A \) and/or \( i = B \).
Step 2. If \( g^i = d_H \) for \( i = A \) and/or \( i = B \), then (47) cannot hold: its LHS is zero, and its RHS is nonnegative for every piece of the best response functions. Therefore, a necessary condition for (47) is that \( Q_{LH}^i \left(Q^{-i}_i \left(R \right), R^i \right) < d_H, i \in \{A, B\} \), so that \( g^i = Q_{LH}^i, i \in \{A, B\} \) satisfy (46). For these functions (47) is equivalent to

\[
\frac{\partial Q_{LH}^A(Q^A(\mathbf{R}), R^B)}{\partial R^B} \frac{\partial Q_{LH}^A(Q^B(\mathbf{R}), R^A)}{\partial Q^B} > -\frac{\partial Q_{LH}^A(Q^B(\mathbf{R}), R^A)}{\partial R^A},
\]

where each term is strictly negative. Substituting from (54) and (56) yields, after some algebra:

\[
R^B \frac{Q^A - d_L}{Q^B} - d_L - \frac{q_{lh}b^A(d_H - Q^B)}{q_{lh} \exp(\mathbf{R}^B b^A(d_H - Q^B)) + q_{lh}} < -\frac{b^A q_{lh}}{q_{lh} \exp(\mathbf{R}^A b^A (d_H - Q^B)) + q_{lh}}.
\]  \hspace{1cm} (58)

**Case 4.** The equilibrium order quantities \( Q^A(\mathbf{R}) \) and \( Q^B(\mathbf{R}) \) satisfy

\[
d^A_{LL} \left(Q^B(\mathbf{R})\right) < Q^A(\mathbf{R}) = Q^A_{LL} \left(Q^B(\mathbf{R}), R^A\right) \leq d_H, \quad d_L < Q^B(\mathbf{R}) = Q^B_{LL} \left(R^B\right) \leq \left[ Q^A_{LL} \left(Q^B(\mathbf{R})\right) \right].
\]

Step 1. The best response function pieces that satisfy (46) are \( g^A = Q_{LH}^A \) and \( g^B = Q_{LL}^B \). This follows since \( Q_{LL}^B \left(R^B\right) < 0 \) by (53) and \( \partial Q_{LH}^A / \partial R^A < 0 \) by (54), whereas \( d_H \) and \( d_{LL} \) are independent of risk aversion.

Step 2. Since \( Q_{LL}^B \) is constant in \( R^A \), it is clear that firm \( B \) reduces its order quantity as both firms become more risk averse. For \( g^A = Q_{LH}^A \) and \( g^B = Q_{LL}^B \), condition (47) is equivalent to

\[
\frac{b^A q_{lh} \exp(\mathbf{R}^A b^A(d_H - Q^B))}{q_{lh} + q_{lh} \exp(-\mathbf{R}^A b^A(d_H - Q^B))} \left( R^A \left( Q^B - d_L \right) + d_H - Q^B \right) > Q^A - d_L.
\]  \hspace{1cm} (59)

**Case 5.** The equilibrium order quantities \( Q^A(\mathbf{R}) \) and \( Q^B(\mathbf{R}) \) satisfy

\[
d_H \leq Q^A(\mathbf{R}) = Q^A_{HL} \left(Q^B(\mathbf{R}), R^A\right) \leq d_H^A \left(Q^B(\mathbf{R})\right) \quad d_L < Q^B(\mathbf{R}) = Q^B_{M} \left(R^B\right) < d_H.
\]

Step 1. The best response function pieces that satisfy (46) are \( g^B = Q_{M}^B \) for firm \( B \), and for firm \( A \) either \( g^A = d_H \) or \( g^A = Q_{HL}^A \). The function \( d_H^A \) cannot be part of the best response as \( \mathbf{R} \) increases, even if \( Q_{HL}^A \left(Q^B(\mathbf{R}), R^A\right) = d_H^A \left(Q^B(\mathbf{R})\right) \): this is because \( d_H^A \) is constant in \( R^A \) while \( Q_{HL}^A \) decreases in \( R^A \) and \( Q_{M}^B \) decreases in \( R^B \). As shown for case 3 if \( g^A = d_H \) then the
condition (47) cannot hold. Therefore, the only pair that satisfies (46) and for which (47) can possibly hold is \( g^A = Q^A_{HL} \) and \( g^B = Q^B_M \).

Step 2. For \( g^A = Q^A_{HL} \) and \( g^B = Q^B_M \), condition (47) is equivalent to

\[
Q^B_M(R^B) \frac{\partial Q^A_{HL}(Q^{B*}(R), R^A)}{\partial Q^B} > - \frac{\partial Q^A_{HL}(Q^{B*}(R), R^A)}{\partial R^A},
\]

where each term is strictly negative. Substituting from (53) and (55) yields, after some algebra the following necessary condition for (47) to hold:

\[
q_{LH}b^A(Q^{B*} - d_L) \left( 1 - \frac{R^B}{R^A} \right) > \frac{R^B}{R^A} (d_H - d_L) \left[ q_{LL} \exp \left( R^A h^A(d_H - Q^{B*}) \right) + q_{LH} (1 - b^A) \right].
\]

Note that this condition is violated if \( R^A \leq R^B \).

**Case 6.** The equilibrium order quantities \( Q^{A*}(R) \) and \( Q^{B*}(R) \) satisfy

\[
Q^{A*} = d^A_{HH}(Q^{B*}(R)) < Q^A_{HL}(Q^{B*}(R), R^A)
\]

\[
d_L < Q^{B*} = Q^B_M(R^B) < d_H.
\]

Step 1. The best response function pieces that satisfy (46) are \( g^A = d^A_{HH} \) and \( g^B = Q^B_M \).

Step 2. Since \( d^A_{HH} \) is constant in \( R^A \) but strictly decreasing in \( Q^B \), and \( g^B \) is strictly decreasing in \( R^B \), it follows that (47) holds.

**Cases 7-17.** The other equilibrium cases of Proposition 2 cannot satisfy (46) and (47).

In all of these cases, except for case 13 and 15, the best response function pieces that satisfy (46) have \( g^i = d_L \) or \( g^i = d_H \) for at least one of the firms. In any such case, the LHS of (47) is zero whereas the RHS is nonnegative.

In case 13, the best response function pieces that satisfy (46) are \( g^i = d^i_{LH} \), for \( i = A \) and \( i = B \).

Since \( d^i_{LH} \) is constant in \( R^i \), both sides of (47) are zero.

In case 15, the best response function pieces that satisfy (46) are \( g^i = Q^i_{LL} \), for \( i = A \) and \( i = B \).

Since \( Q^i_{LL} \) is constant in \( Q^{-i} \), the LHS of (47) is zero whereas the RHS is nonnegative.

**Proof of Corollary 3.** Based on Corollary 2, when both firms are identical except for their \( C_u^i/C_o^i \) ratio, the firm with the larger ratio orders more at the equilibrium. By inspection of the equilibrium cases specified in Proposition 2 and Theorem 2, cases 1, 4 and 6 apply for the firm with the larger equilibrium order, and case 2 applies for the firm with the smaller equilibrium order.

This establishes Parts 1 and 2(a) of the Corollary.

Part 2(b). For equilibrium case 3 it is a priori not clear from Theorem 2 which of the firms increases its order in response to higher risk aversion. The following argument shows that it cannot be the firm with the higher \( C_u^i/C_o^i \) ratio. In particular, the condition (27) cannot hold in this case.
For concreteness, let \( i = A \) be the firm that increases its order in response to higher risk aversion. In light of Corollary 2, we need to show that firm \( A \) must have the smaller equilibrium order.

From (58) in the proof of Theorem 2, condition (27) for firm \( A \) to increase its order from a case 3 equilibrium is equivalent to:

\[
R^B Q^{A*} - d_L - \frac{q_{LL} b^A (d_H - Q^{B*})}{q_{LL} \exp(R^B b^A (d_H - Q^{B*})) + q_{LH}} < b^A q_{LH} \frac{b^A q_{LH}}{q_{LL} \exp(R^B b^A (d_H - Q^{B*})) + q_{LH}}. \tag{61}
\]

With \( R^i = R, K^i = C^i_u + C^i_o = K \) and \( b^i = b \), this condition simplifies to

\[
\frac{Q^{A*} - d_L - \frac{q_{LL} b(d_H - Q^{B*})}{q_{LL} \exp(Rb(d_H - Q^{B*})) + q_{LH}}}{Q^{B*} - d_L - \frac{q_{LL} b(d_H - Q^{A*})}{q_{LL} \exp(Rb(d_H - Q^{A*})) + q_{LH}}} < b \frac{q_{LH}}{q_{LL} \exp(Rb(d_H - Q^{B*})) + q_{LH}}. \tag{62}
\]

We show that the above condition does not hold if the order quantity of Firm \( A \) is larger than the order quantity of Firm \( B \) at the equilibrium, \( Q^{A*} > Q^{B*} \). Before attempting to show the claim, note that condition (62) does not hold for the case that both firms’ order quantities are identical (i.e. in this case, both firms have the same profitability ratio), since the LHS of condition (62) is 1 and the RHS is less than 1, so the condition is violated.

Suppose Firm \( A \)’s order quantity is larger than Firm \( B \)’s order quantity at the equilibrium, \( Q^{A*} > Q^{B*} \). We know there exist \( \epsilon = \sqrt{0.5d} \), where \( d \) is the distance between point \( (Q^{A*}, Q^{B*}) \) and its orthogonal projection \( (Q^A_p, Q^B_p) \) on diagonal line \( (Q^A = Q^B) \), such that

1) \( Q^{A*} = Q^A_p + \epsilon \), 2) \( Q^{B*} = Q^B_p - \epsilon \) and 3) \( Q^A_p = Q^B_p \)

We know that condition (62), does not hold for the following order quantities: \( (Q^A_p, Q^B_p) \) since \( Q^A_p = Q^B_p \). Define the following function:

\[
g(Q^i, Q^i) = Q^i - d_L - \frac{q_{LH} b (d_H - Q^i)}{q_{LL} \exp(Rb(d_H - Q^i)) + q_{LH}}
\]

Note that if \( d_L + b^i (d_H - Q^i) < Q^i < d_H \), for \( i = A, B \), then

\[
g(Q^i, Q^i) = Q^i - d_L - \frac{q_{LH} b (d_H - Q^i)}{q_{LL} \exp(Rb(d_H - Q^i)) + q_{LH}} > Q^i - d_L - b (d_H - Q^i) > 0.
\]

Therefore, \( g(Q^i, Q^i) > 0 \) for \( Q^i, Q^i = (Q^A_p + \epsilon, Q^B_p - \epsilon) \) and \( Q^i, Q^i = (Q^B_p - \epsilon, Q^A_p + \epsilon) \).
We have:

\[
g(Q_p^A + \epsilon, Q_p^B - \epsilon) = Q_p^A + \epsilon - d_L - \frac{q_{LH}b(d_H - Q_p^B + \epsilon)}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LH}}
\]

\[
= Q_p^A - d_L - \frac{q_{LH}b(d_H - Q_p^B)}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LH}} + \epsilon \left(1 - \frac{q_{LH}b}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LH}}\right)
\]

\[
> g(Q_p^A, Q_p^B) + \epsilon \left(1 - \frac{q_{LH}b}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LH}}\right),
\]

and

\[
g(Q_p^B - \epsilon, Q_p^A + \epsilon) = Q_p^B - \epsilon - d_L - \frac{q_{LH}b(d_H - Q_p^A - \epsilon)}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LH}}
\]

\[
= Q_p^B - d_L - \frac{q_{LH}b(d_H - Q_p^A)}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LH}} - \epsilon \left(1 - \frac{q_{LH}b}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LH}}\right)
\]

\[
< g(Q_p^B, Q_p^A) - \epsilon \left(1 - \frac{q_{LH}b}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LH}}\right).
\]

When Firm A orders \(Q_p^A + \epsilon\) and Firm B orders \(Q_p^B - \epsilon\) the RHS of condition (62) is still less than 1, but we show that the LHS becomes bigger than 1. The LHS of condition (62) can be written as \(g(Q_p^A + \epsilon, Q_p^B - \epsilon) / g(Q_p^B - \epsilon, Q_p^A + \epsilon)\). On the other hand we have:

\[
\frac{g(Q_p^A + \epsilon, Q_p^B - \epsilon)}{g(Q_p^B - \epsilon, Q_p^A + \epsilon)} > \frac{g(Q_p^A, Q_p^B) + \epsilon \left(1 - \frac{q_{LH}b}{q_{LL}\exp(Rb(d_H - Q_p^B + \epsilon)) + q_{LH}}\right)}{g(Q_p^B, Q_p^A) - \epsilon \left(1 - \frac{q_{LH}b}{q_{LL}\exp(Rb(d_H - Q_p^A - \epsilon)) + q_{LH}}\right)} > 1
\]

where the first inequality holds because of inequalities (63) and (64), the second inequality holds since \(Q_p^A = Q_p^B\), and therefore \(g(Q_p^A, Q_p^B) = g(Q_p^B, Q_p^A)\).

Therefore, if firm A increases its order in response to higher risk aversion, it must be the firm with the smaller initial equilibrium order, and therefore the firm with the smaller \(C_u^i/C_s^i\) ratio.

Part 3. That Case 5 never holds follows since the necessary condition (60) is violated when \(R^A = R^B\). See the proof of Theorem 2.

**Proof of Corollary 4.** Based on Corollary 2, when both firms are identical except for their risk aversion parameters, the firm with the lower risk aversion parameter orders more at the equilibrium. By inspection of the equilibrium cases specified in Proposition 2 and Theorem 2, cases 1, 4 and 6
apply for the firm with the larger equilibrium order, and case 2 applies for the firm with the smaller equilibrium order. This establishes Parts 1 and 2(a) of the Corollary.

Part 2(b). For equilibrium case 3 it is a priori not clear from Theorem 2 which of the firms increases its order in response to higher risk aversion. The following argument shows that it cannot be the firm with the initially lower \( R^i \). In particular, the condition (27) cannot hold in this case. For concreteness, let \( i = A \) be the firm that increases its order in response to higher risk aversion. In light of Corollary 2, we need to show that firm \( A \) must have the larger initial risk aversion parameter. From (58) in the proof of Theorem 2, with \( b^i = b \), condition (27) for firm \( A \) to increase its order from a case 3 equilibrium is equivalent to:

\[
\frac{R^B Q^A_x - d_L - \frac{q_{LL} b (d_H - Q^B_x)}{q_{LL} \exp(R^A b (d_H - Q^B_x)) + q_{LL}}}{R^A Q^B_x - d_L - \frac{q_{LL} b (d_H - Q^A_x)}{q_{LL} \exp(R^A b (d_H - Q^A_x)) + q_{LL}}} < \frac{b q_{LL}}{q_{LL} \exp(R^A b (d_H - Q^B_x)) + q_{LL}}. \quad (65)
\]

We show that the above condition does not hold if the order quantity of Firm \( A \) is larger than the order quantity of Firm \( B \) at the equilibrium, \( Q^A_x > Q^B_x \). We know there exist \( \epsilon = \sqrt{0.5}d \), where \( d \) is the distance between point \( (Q^A_x, Q^B_x) \) and its orthogonal projection \( (Q^A_p, Q^B_p) \) on diagonal line \( (Q^A = Q^B) \), such that

1) \( Q^A_x = Q^A_p + \epsilon \), 2) \( Q^B_x = Q^B_p - \epsilon \) and 3) \( Q^A_p = Q^B_p = Q_p \).

Note that if \( d_L + b^i (d_H - Q^i) < Q^i < d_H \), for \( i = A, B \), then

\[
Q^i - d_L - \frac{q_{LL} b (d_H - Q^i)}{q_{LL} \exp(R^A b (d_H - Q^i)) + q_{LL}} > Q^i - d_L - b (d_H - Q^i) > 0.
\]

We show inequality (65) does not hold, when Firm \( A \) orders \( Q^A_x = Q^A_p + \epsilon \) and firm \( B \) orders \( Q^B_x = Q^B_p - \epsilon \). In light of Corollary 2, this can happen only if \( R^A < R^B \). In this case the RHS of the condition (65) is still less than 1, but we claim that the LHS is larger than 1:

\[
\frac{R^B Q^A_p + \epsilon - d_L - \frac{q_{LL} b (d_H - Q^B_p + \epsilon)}{q_{LL} \exp(R^A b (d_H - Q^B_p + \epsilon)) + q_{LL}}}{R^A Q^B_p - \epsilon - d_L - \frac{q_{LL} b (d_H - Q^A_p - \epsilon)}{q_{LL} \exp(R^A b (d_H - Q^A_p - \epsilon)) + q_{LL}}} > 1. \quad (66)
\]

for simplicity we replace \( Q^A_p \) and \( Q^B_p \) with \( Q_p \). Therefore, we have:

\[
\frac{R^B}{R^A} \left( Q_p + \epsilon - d_L - \frac{q_{LL} b (d_H - Q_p + \epsilon)}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LL}} \right) > Q_p - \epsilon - d_L - \frac{q_{LL} b (d_H - Q_p - \epsilon)}{q_{LL} \exp(R^A b (d_H - Q_p - \epsilon)) + q_{LL}}.
\]
Rearranging the above inequality yields:

\[
\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) \epsilon - \frac{R^B}{R^A} \left( \frac{q_{LH} b (d_H - Q_p)}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) > 0
\]  \hspace{1cm} (67)

\[
- \beta e \left[ \frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) > \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} \right] > 0
\]

To establish (67), it suffice to show that the following inequality holds:

\[
\text{LHS of inequality (67)} > \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon - \frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right) b (d_H - Q_p).
\]  \hspace{1cm} (68)

This is because the RHS of inequality (68) is larger than 0:

\[
\left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon - \frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right) b (d_H - Q_p) > \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L - b (d_H - Q_p)) > 0,
\]

because \( \frac{R^B}{R^A} - 1 > 0 \) (since \( R^A < R^B \)) and \( Q_p \geq d_L + b (d_H - Q_p) \) (based on Proposition 2, both firms order quantities are between their own minimum demand plus spillover and their own maximum demand).

Therefore we now focus on showing that inequality (68) holds. We break inequality (68) into two inequalities i.e. Inequality A and B where the LHS (RHS) of Inequality A plus the LHS (RHS) of Inequality B equals the LHS (RHS) of inequality (68). Therefore, showing inequalities A and B holds, proves that inequality (68) holds. We define the inequality A and B as follows:

Inequality A:

\[
- \beta e \left[ \frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) > \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}} \right] > \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) \epsilon
\]

and Inequality B:

\[
- \frac{R^B}{R^A} \left( \frac{q_{LH}}{q_{LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{LH}} \right) > \frac{q_{LH}}{q_{LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{LH}}
\]

\[
> - \frac{q_{LH}}{q_{LL} + q_{LH}} \left( \frac{R^B}{R^A} - 1 \right).
\]

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First we show that inequality A holds. Since

\[
\left( \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \right) < 1,
\]

\[
\frac{q_{ LH}}{q_{ LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{ LH}} < 1.
\]

therefore we have:

\[
-b \left[ \frac{R^B}{R^A} \left( \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \right) + \frac{q_{ LH}}{q_{ LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{ LH}} \right] > \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) \epsilon - b \left[ \frac{R^B}{R^A} + 1 \right] = \left( \frac{R^B}{R^A} - 1 \right) (Q_p - d_L) + \left( \frac{R^B}{R^A} + 1 \right) (1 - b) \epsilon
\]

which proves that inequality A holds.

Next we show that inequality B holds. If \( R^A b (d_H - Q_p + \epsilon) \geq R^B b (d_H - Q_p - \epsilon) \), we have:

\[
\frac{R^B}{R^A} \left( \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \right) - \frac{q_{ LH}}{q_{ LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{ LH}} < \frac{R^B}{R^A} \left( \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \right) - \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} < \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \left( \frac{R^B}{R^A} - 1 \right)
\]

which shows that inequality B holds in this case. However, if \( R^A b (d_H - Q_p + \epsilon) < R^B b (d_H - Q_p - \epsilon) \), we have:

\[
\frac{R^B}{R^A} \left( \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \right) - \frac{q_{ LH}}{q_{ LL} \exp(R^B b (d_H - Q_p - \epsilon)) + q_{ LH}} < \left( \frac{R^B}{R^A} - 1 \right) \left( \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \right) - \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} < \frac{q_{ LH}}{q_{ LL} \exp(R^A b (d_H - Q_p + \epsilon)) + q_{ LH}} \left( \frac{R^B}{R^A} - 1 \right).
\]
which also shows that inequality B holds in this case. Since Inequality A and B hold, it follows that inequality (68) holds which establishes (66). If firm A increases its order in response to higher risk aversion, it must be the firm with the smaller initial equilibrium order, and therefore the firm with the higher initial risk aversion rate.

Part 3. By inspection of the equilibrium case 5 specified in Proposition 2 and Theorem 2, case 5 applies for the firm with the larger equilibrium order, which is the firm with the smaller risk aversion parameter by Corollary 2. Say it is firm A and $R^B/R^A > 1$. However, the necessary condition (60) is violated when $R^B/R^A > 1$. See the proof of Theorem 2.

Proof of Corollary 5. Based on Corollary 2, when both firms are identical except for their spillover fractions, the firm with the larger spillover fractions orders at least as much as its rival at the equilibrium. By inspection of the equilibrium cases specified in Proposition 2 and Theorem 2, it is a priori not clear whether these cases can occur, and if so, whether the firm with the lower or the one with the higher spillover fraction increases its order. The proof clarifies this point.

Part 1(a). Case 1. The following argument shows that the equilibrium conditions for case 1 of Proposition 2 can only hold for the firm with the lower spillover fraction and equilibrium order. Consider case 1 of Theorem 2, where firm A orders the minimum demand plus spillover and firm B orders between the minimum demand and the minimum demand plus spillover. Firm A must have smaller spillover rate to increase its order quantity. So the claim is $b^A < b^B$. We prove it by contradiction. Suppose that $b^A \geq b^B$. Therefore by Corollary 2, we have $Q^A \geq Q^B$. Based on Proposition 2 the following inequality must hold for Firm B to order between the minimum demand and the minimum demand plus spillover:

$$\frac{C^B_u}{C^B_o} < \frac{q_{LL}}{1-q_{LL}} e^{R^B b^B (d_H - Q^A)}.$$

However, since both firms are identical except for spillover rate we must have:

$$\frac{C^A_u}{C^A_o} = \frac{C^B_u}{C^B_o} \leq \frac{q_{LL}}{1-q_{LL}} e^{R^B b^B (d_H - Q^A)} \leq \frac{q_{LL}}{1-q_{LL}} e^{R^A b^A (d_H - Q^B)},$$

where the second inequality holds because $b^A \geq b^B$ and $Q^A \geq Q^B$, which means that $b^B (d_H - Q^A) \leq b^A (d_H - Q^B)$.

However, having the following inequality

$$\frac{C^A_u}{C^A_o} \leq \frac{q_{LL}}{1-q_{LL}} e^{R^A b^A (d_H - Q^B)},$$

contradicts the condition in Proposition 2 for firm A in case 1 to order minimum demand plus spillover.
**Part 1(b).** Case 4. Consider case 4 where firm A orders between the minimum demand plus spillover and the maximum demand and Firm B order between the minimum demand and the minimum demand plus spillover:

\[
Q^A = d_L + b^A(d_H - Q^B) + \frac{\ln \left( \left( \frac{C_u}{C_o} \right) \left( \frac{1-q_{LL} - q_{HH}}{q_{LL} e^{R^A b^A (d_H - Q^B) + q_{LL} H}} \right) \right)}{R^A} \quad \text{and} \quad Q^B = d_L + \frac{\ln \left( \left( \frac{C_u}{C_o} \right) \left( \frac{1-q_{LL}}{q_{LL} \text{ln} \cdot \cdot \cdot} \right) \right)}{R^B}.
\]

The following argument shows that the equilibrium conditions for case 4 of Proposition 2 can only hold for the firm with the lower spillover fraction and equilibrium order, so the claim is \( b^A < b^B \).

We prove the claim by contradiction. Suppose that \( b^A \geq b^B \). Based on Proposition 2, for Firm B to order between the minimum demand and the minimum demand plus spillover, we must have:

\[
\frac{q_{LL}}{1-q_{LL}} < \frac{C_u}{C_o} \leq \frac{q_{LL}}{1-q_{LL}} e^{R^B b^B (d_H - Q^A)}.
\]

Since both firms are identical except for their spillover rate we must have:

\[
\frac{q_{LL}}{1-q_{LL}} < \frac{C_u}{C_o} = \frac{q_{LL}}{1-q_{LL}} e^{R^A b^A (d_H - Q^A)} \leq \frac{q_{LL}}{1-q_{LL}} e^{R^B b^B (d_H - Q^A)},
\]

where the last inequality holds because \( b^A \geq b^B \) and \( Q^A \geq Q^B \), which means that \( b^B (d_H - Q^A) \leq b^A (d_H - Q^B) \). However, based on Lemma 4, if the following inequality

\[
\frac{q_{LL}}{1-q_{LL}} < \frac{C_u}{C_o} \leq \frac{q_{LL}}{1-q_{LL}} e^{R^A b^A (d_H - Q^A)}
\]

holds, Firm A’s order quantity must be between the minimum demand and the minimum demand plus spillover, which is a contradiction.

**Part 2.** (Case 2). Consider case 2 where firm A orders the minimum demand plus spillover and firm B orders between the minimum demand plus spillover and the maximum demand:

\[
Q^A = d_L + b^A(d_H - Q^B) \quad \text{and} \quad Q^A < Q_X = \frac{(1-b^A)d_L + b^A(1-b^B)d_H}{1-b^A b^B}
\]

\[
Q^B = d_L + b^B(d_H - Q^A) + \frac{\ln \left( \left( \frac{C_u}{C_o} \right) \left( \frac{1-q_{LL} - q_{HH}}{q_{LL} e^{R^B b^B (d_H - Q^A) + q_{HH}}} \right) \right)}{R^B}.
\]

The following argument shows that the equilibrium conditions for case 2 of Proposition 2 can only hold for the firm with the higher spillover fraction and equilibrium order. We prove the claim by contradiction. Suppose that \( b^B \geq b^A \). Based on Proposition 2, if Firm A orders the minimum demand plus spillover in equilibrium, we must have:

\[
\frac{C_u}{C_o} \leq \frac{q_{LL} e^{R^A b^A (d_H - d_L) + q_{HH}}}{1-q_{LL} - q_{HH}}.
\]
and since both firms are identical except for their spillover rate (based on our assumption: $b^B \geq b^A$), we must have:

$$\frac{C^B_u}{C^B_o} = \frac{C^A_u}{C^A_o} \leq \frac{q_{LL} e^{\bar{R}^A(d_H - d_L)} + q_{LH} e^{\bar{R}^A(d_H - d_L)} + q_{HL} e^{\bar{R}^B(b_H - d_L) + q_{LH}}}{1 - q_{LL} - q_{LH}} \leq \frac{q_{LL} e^{\bar{R}^B(d_H - d_L) + q_{LH}}}{1 - q_{LL} - q_{LH}}$$

where the second inequality holds since $b^B \geq b^A$. However, based on Lemma 4 the above inequality means that Firm B’s order quantity can be at most equal to minimum demand plus spillover, which is a contradiction.

**Part 3.** That Case 5 never holds follows since the necessary condition (60) is violated when $R^A = R^B$. See the proof of Theorem 2.

Case 6. Consider case 6, where firm A orders the maximum demand plus spillover and firm B adopts monopoly strategy and orders between the minimum demand and maximum demand:

$$Q^A = d_H + b^A(d_H - Q^B) \text{ and } Q^B = d_L + \ln\left(\frac{C^B_u}{C^B_o} \left(\frac{1 - q_L}{q_L}\right)\right).$$

The following argument shows that the equilibrium conditions for case 6 of Proposition 2 cannot hold. Based on Proposition 2 to have such an equilibrium, the following conditions must hold for firms A and B:

$$\frac{C^A_u}{C^A_o} > \frac{q_{LL} e^{\bar{R}^A(d_H - d_L) + b^A(d_H - Q^B)} + q_{LH} e^{\bar{R}^A(d_H - d_L)} + q_{HL} e^{\bar{R}^B(d_H - Q^B)}}{1 - q_{LL} - q_{LH} - q_{HL}}, \quad (69)$$

$$\frac{C^B_u}{C^B_o} < \frac{q_{LL} e^{\bar{R}^B(d_H - d_L)}}{(1 - q_L)}. \quad (70)$$

We show that conditions (69) and (70) are mutually exclusive. Since both firms are identical except for spillover rate, i.e. $C^A_u/C^A_o = C^B_u/C^B_o$, and $\bar{R}^A = \bar{R}^B$, it suffices to show that the RHS of inequality (69) is larger than the RHS of inequality (70).

Note that the RHS of inequality (69) is a decreasing function of $Q^B$, therefore inequality (69) must hold for $Q^B = d_H$ :

$$\frac{C^A_u}{C^A_o} > \frac{e^{\bar{R}^A(d_H - d_L)}(q_{LL} + q_{LH}) + q_{HL}}{1 - q_{LL} - q_{LH} - q_{HL}}. \quad (71)$$

However, it is straightforward to see that the following inequality holds

$$\frac{e^{\bar{R}^A(d_H - d_L)}(q_{LL} + q_{LH}) + q_{HL}}{1 - q_{LL} - q_{LH} - q_{HL}} = \frac{e^{\bar{R}^A(d_H - d_L)}q_L + q_{HL}}{1 - q_L - q_{HL}} > \frac{q_{LL} e^{\bar{R}^A(d_H - d_L)}}{(1 - q_L)},$$

where $q_{LL} + q_{LH} = q_L$. Therefore we have shown that conditions (69) and (70) are mutually exclusive. Therefore when both firms are identical except for their spillover rate, case 6 cannot happen.
Proof of Theorem 3. Let $A = i$ and $B = j$. Suppose in perfect positive correlation case, the total demand of each firm has only two points: minimum demand and maximum demand plus spillover. Based the best response characterization in (32), there are six possibilities for the location of equilibrium relative to the demand points:

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>$Q^i$</th>
<th>$Q^j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$d_L$</td>
<td>$d_L$</td>
</tr>
<tr>
<td>2</td>
<td>$d_L + \frac{1}{R} \ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)$</td>
<td>$d_L$</td>
</tr>
<tr>
<td>3</td>
<td>$d_H + b^i(d_H - d_L)$</td>
<td>$d_L + \frac{1}{R} \ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right) &lt; d_H$</td>
</tr>
<tr>
<td>4</td>
<td>$d_L + \frac{1}{R} \ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)$</td>
<td>$d_L + \frac{1}{R} \ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)$</td>
</tr>
<tr>
<td>5</td>
<td>$d_L + b^i(d_H - Q^j)$</td>
<td>$d_L + \frac{1}{R} \ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)$</td>
</tr>
<tr>
<td>6</td>
<td>$d_H$</td>
<td>$d_H$</td>
</tr>
</tbody>
</table>

Note that except for equilibrium 6, in the other equilibria neither firm’s equilibrium order quantity depends on its competitor’s order quantity. Therefore the own risk aversion effect is the only factor that has an impact on each firm’s order quantity. As a result both firms either reduce or do not change their order quantities as both become more risk averse. For equilibrium 6, it is clear by inspection that firm $i$ increases its order quantity as both firms become more risk averse. In particular firm $j$ reduces its order quantity as both firms become more risk averse since the effect of its own risk aversion is the only factor that has an impact on its order quantity and hence firm $i$ increases its order quantity. Since we have closed form solution for the equilibrium, the conditions for equilibrium type 6 follow immediately.

Proof of Theorem 4. Let $A = i$ and $B = j$. In perfect negative correlation case, the total demand of each firm has only two points: minimum demand plus spillover and maximum demand. Based the best response characterization in (33), there are six possibilities for the location of equilibrium relative to the demand points:

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>$Q^i$</th>
<th>$Q^j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$d_L + b^j(d_H - Q^j) = \frac{d_L(1-b^j) + d_H b^j(1-b^j)}{1-b^j b^j}$</td>
<td>$d_L + b^j(d_H - Q^j) = \frac{d_L(1-b^j) + d_H b^j(1-b^j)}{1-b^j b^j}$</td>
</tr>
<tr>
<td>2</td>
<td>$d_L + b^j(d_H - Q^j)$</td>
<td>$d_L + b^j(d_H - Q^j) + \frac{\ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)}{R}$</td>
</tr>
<tr>
<td>3</td>
<td>$d_L + b^j(d_H - Q^j) + \frac{\ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)}{R} &lt; d_H$</td>
<td>$d_L + b^j(d_H - Q^j) + \frac{\ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)}{R} &lt; d_H$</td>
</tr>
<tr>
<td>4</td>
<td>$d_L$</td>
<td>$d_H$</td>
</tr>
<tr>
<td>5</td>
<td>$d_L + \frac{\ln \left( \frac{C_j}{C_i} \left( \frac{1-q_L}{q_L} \right) \right)}{R}$</td>
<td>$d_H$</td>
</tr>
<tr>
<td>6</td>
<td>$d_H$</td>
<td>$d_H$</td>
</tr>
</tbody>
</table>
Note that except for equilibriums 2 and 3, in the other equilibria neither firm’s equilibrium order quantity depends on its competitor’s order quantity. Therefore the own risk aversion effect is the only factor that has an impact on each firm’s order quantity. As a result both firms either reduce or do not change their order quantities as both become more risk averse. Next we focus on equilibriums 2 and 3 respectively.

Part 1. For concreteness suppose that $i = A$ and $j = B$. It is clear by inspection that firm $A$ increases its order quantity as both firms become more risk averse, if the order quantities of the of firms $A$ and $B$ are as follows:

$$Q^A = d_L + b^A(d_H - Q^B) = \frac{d_L (1 - b^A) + b^A d_H (1 - b^B) - b^A \ln \left[ \frac{C^B}{C^A} \left( \frac{1}{q_L} \right) \right]}{1 - b^A b^B},$$

$$Q^B = d_L + b^B(d_H - Q^A) + \frac{\ln \left[ \frac{C^A}{C^B} \left( \frac{1}{q_L} \right) \right]}{R^B} = \frac{d_L (1 - b^B) + b^B d_H (1 - b^A) + \ln \left[ \frac{C^B}{C^A} \left( \frac{1}{q_L} \right) \right]}{1 - b^A b^B}.$$

Since we have closed form solution for the equilibrium, the conditions for equilibrium type 2 follow immediately from the best response characterization.

Part 2. For concreteness suppose that $i = A$ and $j = B$. If firm $A$ and B’s order quantity at the equilibrium are as follows:

$$Q^A = d_L + b^A(d_H - Q^B) + \frac{\ln \left[ \frac{C^A}{C^B} \left( \frac{1}{q_L} \right) \right]}{R^A},$$

$$Q^B = d_L + b^B(d_H - Q^A) + \frac{\ln \left[ \frac{C^A}{C^B} \left( \frac{1}{q_L} \right) \right]}{R^B},$$

then firm $A$ increases its order quantity as both firms become more risk averse if the following condition holds:

$$\frac{\partial f^A(Q^B, R^A)}{\partial Q^B} \times \frac{df^B(Q^A, R^B)}{dR^B} + \frac{df^A(Q^B, R^A)}{dR^A} > 0,$$

where

$$b^A \left( \ln \left[ \frac{C^B}{C^A} \left( \frac{1}{q_L} \right) \right] \frac{R^B}{R^B R^B} \right) - \ln \left[ \frac{C^A}{C^B} \left( \frac{1}{q_L} \right) \right] \frac{R^B}{R^A R^A} > 0,$$

or

$$\frac{C^B}{C^A} > \frac{q_L}{(1 - q_L)} \left[ \frac{C^A}{C^B} \left( \frac{1}{q_L} \right) \right] \frac{R^B}{R^A R^A}.$$

To show the desired condition for the equilibrium, again since we have closed form solution for the equilibrium, the conditions for equilibrium type 3 follow immediately from the best response characterization.■