Asset Pricing with Structural Uncertainty and Structural Disagreement

Daniel Andrei* Bruce Carlin† Michael Hasler‡

March 11, 2017

Abstract

We study a pure exchange economy with incomplete information in which two agents are uncertain and disagree about the length of business cycles. That is, the agents do not question whether the economy is growing or not, but instead continuously estimate how long economic cycles will last. Learning about persistence generates high and persistent stock-return volatility mostly during recessions, but also (to a smaller extent) during economic booms. Disagreement among agents fluctuates and earns a risk premium. A clear risk-return tradeoff appears only when conditioning on the sign and magnitude of disagreement. We confirm these predictions empirically.

Keywords: Learning, Uncertainty, Disagreement, Volatility, Risk Premium
JEL Classification: D51, D83, G12, G14

We would like to thank Tony Berrada, Mike Chernov, Julien Cujean, Alexander David, Jerome Detemple, Bernard Dumas, Andrea Eisfeldt, Barney Hartman-Glaser, Burton Hollifield, Julien Hugonnier, Arvind Krishnamurthy, Lars-Alexander Kuehn, Ali Lazrak, Francis Longstaff, Hanno Lustig, Emilio Osambela, Monika Piazzesi, Rodolfo Prieto, Nick Roussanov, Martin Schneider, Ken Singleton, Pascal St.-Amour, Wei Xiong, and Hongjun Yan for their useful advice. We would also like to acknowledge comments from conference and seminar participants at the SFI meeting in Gerzensee, 4th International Forum on Long-Term Risks, UCLA Anderson, 2014 Mathematical Finance Days, 2014 SFS Finance Cavalcade, 2014 WFC, 2014 SITE Summer Workshop, 2014 EFA meeting, 2014 NFA meeting, CMU Tepper, the New York Fed, 2016 International Conference on Capital Markets, and 2016 SGF Conference. Financial support from the Swiss Finance Institute, NCCR FINRISK of the Swiss National Science Foundation, UCLA, and the University of Toronto is gratefully acknowledged.

*UCLA, Anderson School of Management, 110 Westwood Plaza, Suite C420, Los Angeles, CA 90095, USA, daniel.andrei@anderson.ucla.edu, danielandrei.info

†UCLA, Anderson School of Management, 110 Westwood Plaza, Suite C413, Los Angeles, CA 90095, USA, bruce.carlin@anderson.ucla.edu, www.anderson.ucla.edu/finance/faculty/carlin

‡University of Toronto, Rotman School of Management, 105 St. George Street, Suite 431, Toronto, ON, M5S 3E6, Canada, michael.hasler@rotman.utoronto.ca, www.rotman.utoronto.ca/faculty/hasler

1 Introduction

Asset return volatility is known to be excessively high relative to the volatility of fundamentals (Shiller, 1981), to be time-varying and predictable (Engle, 1982), to increase during recessions (Schwert, 1989) but also during periods of rapid technological progress (Pastor and Veronesi, 2006). So far, no consensus has been reached as to which model can simultaneously explain all of these empirical regularities. The purpose of this paper is to offer such a model.

We study a pure exchange economy with incomplete information in which two agents are uncertain about the length of the business cycle. That is, agents do not question whether the economy is growing or not—the usual learning exercise studied in the literature—but instead continuously wonder how long economic cycles will last. For example, during the Great Recession of 2007-08 there was no debate about whether there was an economic downturn, rather economists were uncertain and disagreed about when recovery would take place. In this paper, we show that *learning about the persistence* of economic growth differs from learning about economic growth itself and helps account for a wide range of empirical observations regarding stock-return volatility.

The first testable implication is that learning about persistence generates time-varying uncertainty and persistent stock market volatility. The quantity responsible for this, which we call *structural uncertainty*, is defined as the product of two variables: the uncertainty about growth persistence and the difference between the growth rate and its long-term mean. Intuitively, structural uncertainty is a function of how much uncertainty is perceived by agents and the degree to which the economy is in an expansion or contraction. This implies that the way in which structural uncertainty impacts the price of risk, risk premia, and the volatility of stock returns depends on the growth rate of the economy.

When the economy is going through recessions or booms, agents face significant structural uncertainty, which increases stock return volatility. This is more severe during recessions, when bad news worsens the agents' forecasts of future growth and increases the persistence (riskiness) of future shocks. We show that there exists a "hockey-stick"-shaped relationship between volatility and economic growth that rationalizes the high levels of volatility observed particularly during recessions (Schwert, 1989; Patton and Timmermann, 2010; Barinov, 2014), but also the less severe levels of volatility during booms such as the Nasdaq bubble in the late 1990's (Pastor and Veronesi, 2006).

Because agents use different sources of information to estimate the length of business cycles, they exhibit time-varying disagreement about persistence in the economy. Based on this, an additional quantity that affects asset prices, which we call *structural disagreement*,

is defined as the product of two quantities: the degree to which agents' estimates of the mean-reversion speed of the fundamental diverge from each other and the difference between the fundamental and its long-term mean. Since structural disagreement is enhanced during booms and recessions, this yields a second testable prediction: structural disagreement generates fluctuations in the agents' consumption shares and thereby impacts the market price of risk in the economy. When the economy goes through booms or recessions, structural disagreement is large and the agent with the least favorable economic outlook perceives the risky asset as a bad hedge against fluctuations in her consumption share. Consequently, she requires a large risk premium for holding the risky asset. Through this mechanism, structural disagreement induces fluctuations in the risk premium in the economy.

Taken together, these two predictions show that the risk-return tradeoff in our economy crucially depends on the magnitude and the sign of structural disagreement (which, according to the above definition, can be both positive and negative). Thus, the third testable prediction of our model is that the risk-return tradeoff becomes apparent only when structural disagreement is taken into account. This can rationalize the inconclusive findings in the literature about the sign and the significance of the relation between risk premium and volatility (Glosten, Jagannathan, and Runkle, 1993; Brandt and Kang, 2004).

We test these three predictions with S&P 500 returns and U.S. output growth data, from which we build proxies for volatility, risk premium, structural uncertainty, and structural disagreement. We find that indeed the volatility of stock returns features a hockey-stick pattern, being higher in recessions. We then document a clear relationship between structural disagreement and the risk premium, as predicted by the model. Finally, we explore the risk-return tradeoff and find that it is indeed dependent on the sign of structural disagreement. Overall, these empirical results generally support our theoretical model of learning and disagreement about the persistence of economic growth.

In addition to these results, our paper provides a theoretical foundation for GARCH. The persistence of volatility has been described extensively in the empirical literature, but there is a paucity of theoretical explanations.[1] The explanation that we provide here is based on structural uncertainty, which arises endogenously from learning about growth persistence.[2]

[1] A few preference-based foundations for volatility clustering are provided by Campbell and Cochrane (1999), Barberis, Huang, and Santos (2001), and McQueen and Vorkink (2004). See also Osambela (2015).

[2] Timmermann (2001) shows that existence of structural breaks in the fundamental process induces volatility clustering. David (1997) and Veronesi (1999) show that, when the fundamental follows a Markov-switching process, learning implies volatility clustering. A commonality between our work and theirs is that agents have imperfect information about the economic model, although our model does not feature structural breaks or a finite number of states. See also Collin-Dufresne, Johannes, and Lochstoer (2015), who show that parameter learning generates long-lasting risks when a representative agent has a preference for early resolution of uncertainty.

Finally, our work rationalizes the empirical findings in Carlin, Longstaff, and Matoba (2014), who analyze how model disagreement in the MBS market affects asset prices. The authors show that there is a risk premium associated with disagreement and that disagreement varies over time. It appears that disagreement rises during periods of large market movements, which is consistent with our idea of structural disagreement.

Our approach departs from the existing asset pricing literature in several ways. First, instead of assuming that agents learn about the unobserved growth rate of the economy, we propose a model whereby agents observe the growth rate but learn about its persistence. As we show in the paper, this type of learning is associated with time-varying uncertainty, which contrasts with learning models with similar structures[3] and is empirically plausible (Jurado, Ludvigson, and Ng, 2015).[4] Furthermore, learning about persistence endogenously generates higher volatility when the economy goes through booms and recessions, and induces an asymmetric response to news in good versus bad times, features responsible for the hockey-stick pattern in volatility. All endogenous quantities in our economy, including volatility and risk premia, are now substantially driven by the level of economic growth and are higher especially during recessions. This view is different from other models of learning in which uncertainty is higher when agents perceive the economy to be "in-between" a discrete set of growth states.[5]

Second, we depart from the usual modeling approach to study the effect of heterogeneous beliefs on asset prices. Typically, this literature assumes that agents agree to disagree about the unobservable fundamental—the expected dividend growth—(e.g., Scheinkman and Xiong, 2003; Dumas, Kurshev, and Uppal, 2009).[6] Instead, we assume that the fundamental is publicly observed and agreed upon at all times, but agents disagree about the parameters of the model that govern its dynamics. This setup is motivated by recent empirical evidence of model disagreement in MBS markets: Carlin et al. (2014) document substantial disagree-

[3]Uncertainty is constant in most learning models, a common result in the broad literature of learning (see Detemple, 1986; Gennotte, 1986; Dothan and Feldman, 1986; Brennan and Xia, 2001, among many others). This arises because priors are Gaussian and all variables are normally distributed. In this case, the conditional variance of the unobservable variable—the Bayesian uncertainty—follows a deterministic path and converges to a steady state. Stochastic uncertainty arises with non-Gaussian distributions (see, for instance, Detemple, 1991; David, 1997; Veronesi, 1999).

[4]See also Bloom (2009), Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2014) and Bachmann, Elstner, and Sims (2013).

[5]David (1997); Veronesi (1999, 2000); Cagetti, Hansen, Sargent, and Williams (2002); David and Veronesi (2002); David (2008).

[6]See also Harris and Raviv (1993), Detemple and Murthy (1994), Kandel and Pearson (1995), Zapatero (1998), Basak (2000, 2005), Berrada (2006, 2009), Buraschi and Jiltsov (2006), Li (2007), David (2008), Yan (2008), Xiong and Yan (2010), Cvitanic and Malamud (2011), Cvitanic, Jouini, Malamud, and Napp (2011), Bhamra and Uppal (2013), Buraschi, Trojani, and Vedolin (2014), Buraschi and Whelan (2013), Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2017), Osambela (2015), Baker, Hollifield, and Osambela (2015), and Dumas, Lewis, and Osambela (2016).

ment among Wall Street mortgage dealers about prepayment speed forecasts, although all of the dealers in the survey are large financial institutions having access to all publicly available information and only very little private information.[7]

Our paper focuses on Bayesian (rational) learning and abstracts from other frictions or behavioral biases. As such, our paper is complementary to earlier contributions in the literature in which volatility clustering and other stylized facts about business cycles can arise from heterogeneous beliefs coupled with liquidity constraints (Osambela, 2015), from overconfidence and over-extrapolation (Alti and Tetlock, 2014), or from extrapolative biases (Hirshleifer, Li, and Yu, 2015). As we explain in the paper, most of these features can enrich and reinforce our main results.

The rest of the paper proceeds as follows. Section 2 defines our model and the learning processes that the agents use, characterizes the market equilibrium, and shows how uncertainty and disagreement affect asset prices. In Section 3, we calibrate the model, present our main theoretical predictions, and test them empirically. Section 4 discusses further considerations about the term structure of disagreement implied by our model specification, about overconfidence, and about agents' survival in this economy. Section 5 concludes. The Appendix contains all proofs, checks the accuracy of our numerical approximation, and describes our calibration exercise.

2 A Model of Uncertainty and Disagreement

Consider a pure exchange economy defined over a continuous-time horizon $[0, \infty)$, in which a single consumption good serves as the numéraire. A single risky asset (the *stock*) pays the aggregate consumption stream, δ, which follows the process

$$d\delta_t = \delta_t f_t dt + \delta_t \sigma_\delta dW_t^\delta \tag{1}$$

$$df_t = \lambda_t(\bar{f} - f_t)dt + \sigma_f(\rho dW_t^\delta + \sqrt{1-\rho^2} dW_t^f) \tag{2}$$

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \Phi dW_t^\lambda, \tag{3}$$

where W^δ, W^f, and W^λ are three independent Brownian motions under the objective probability measure $\mathbb{P}$.

The expected consumption growth rate f, which we regard as the *fundamental*, mean-reverts around its long-term mean $\bar{f}$ at speed λ. The mean-reversion speed λ is assumed to fluctuate, as described in (3). This process drives the length of recessions and expansions and can be affected, for instance, by continuous technological change of the economic

[7] See also Patton and Timmermann (2010), Andrade, Crump, Eusepi, and Moench (2017).

environment.[8]

The economy is populated by two agents, A and B, who trade with each other and derive utility from consumption. Each agent chooses a consumption-trading policy to maximize her expected lifetime utility

$$U_i = \mathbb{E}^i \left[\int_0^\infty e^{-\beta t} \frac{c_{it}^{1-\alpha}}{1-\alpha} dt \right], \tag{4}$$

where $\beta > 0$ is the time discount rate, $\alpha > 0$ is the relative risk aversion coefficient, and c_{it} denotes the consumption of agent $i \in \{A, B\}$ at time t. The expectation in (4) depends on agent i's own perception of the economy. The parameters $\bar{f}$, σ_δ, σ_f, ρ, κ, $\bar{\lambda}$, and Φ are commonly known.

2.1 Learning and difference of beliefs

At all times, both agents observe the output δ and the fundamental f. The fundamental can be interpreted as the average (median) forecast of the output growth rate among a large survey of professional forecasters, observed and agreed upon by both agents.

Agents do not observe the mean-reversion speed λ and are therefore tasked with estimating it. This feature distinguishes our model from previous work: in existing models, it is assumed that agents do not observe the fundamental f and have heterogeneous beliefs about it (e.g., Scheinkman and Xiong, 2003; Dumas et al., 2009).[9] Here, both agents publicly observe f and its evolution with no disagreement, but try to estimate its mean-reversion speed λ in order to predict the length of business cycles. The question that agents are facing is how long booms or recessions will last before the economy moves back to its known long-term growth rate $\bar{f}$. For instance, the issue during the Great Recession of 2007-08 was not whether we were in recession—it was pretty clear we were—but how long it would last.

We introduce heterogeneity of beliefs by adopting the "difference-of-opinion" approach (Harris and Raviv, 1993; Kandel and Pearson, 1995). More precisely, suppose that each

[8]The mean-reversion speed can become negative because it follows an Ornstein-Uhlenbeck process. The calibration provided in Table 1, however, imply that the unconditional probability of a negative λ_t is only 0.08%. This ensures that the fundamental is stationary. The stationarity properties of an Ornstein-Uhlenbeck process with stochastic mean-reversion speed are discussed by Benth and Khedher (2016).

[9]Besides long-run behavior, there are other dimensions of parameter uncertainty analyzed in the literature, such as tail events (Liu, Pan, and Wang, 2005) or regime changes (Ju and Miao, 2012). See Collin-Dufresne et al. (2015) for a good survey of the literature. The mean-reversion speed λ, which is the focus of this paper, has spawned serious interest and disagreement among practitioners and academics. It has been at the center of understanding recoveries following economic crises (Reinhart and Rogoff, 2009; Howard, Martin, and Wilson, 2011; Bernanke, 2013) and in studying long-term trends in economic growth (Beeler and Campbell, 2012; Bansal, Kiku, and Yaron, 2012; Summers, 2014; Hamilton, Harris, Hatzius, and West, 2015).

agent receives a different signal about λ:

$$ds_t^A = \phi dW_t^\lambda + \sqrt{1-\phi^2} dW_t^A \tag{5}$$

$$ds_t^B = \phi dW_t^\lambda + \sqrt{1-\phi^2} dW_t^B, \tag{6}$$

where W^A and W^B are two additional, independent Brownian motions. Agents do not learn from each others' behavior, i.e., they do not trust the information source of the other agent. The parameter ϕ defines the level of divergence of opinion: if $\phi = 1$, agents are in perfect agreement and the setup reduces to a representative agent economy; if $0 < \phi < 1$, the two signals are different and the agents will disagree about λ. If $\phi = 0$, neither of the signals is informative and therefore agents are again in perfect agreement.[10]

The filtered mean-reversion speed (*filter*) $\widehat{\lambda}$ and its posterior variance (*uncertainty*) γ are such that λ_t is normally distributed with mean $\widehat{\lambda}_t$ and variance γ_t. Based on (1)-(6), Bayesian learning implies:[11]

$$d\zeta_t = \left(f_t - \frac{1}{2}\sigma_\delta^2\right) dt + \sigma_\delta d\widehat{W}_t^\delta \tag{7}$$

$$df_t = \widehat{\lambda}_t^i(\bar{f} - f_t)dt + \sigma_f \rho d\widehat{W}_t^\delta + \sigma_f\sqrt{1-\rho^2} d\widehat{W}_t^{fi} \tag{8}$$

$$d\widehat{\lambda}_t^i = \kappa(\bar{\lambda} - \widehat{\lambda}_t^i)dt + \frac{\gamma_t}{\sigma_f\sqrt{1-\rho^2}}(\bar{f} - f_t)d\widehat{W}_t^{fi} + \phi\Phi d\widehat{W}_t^{si}, \tag{9}$$

where $\zeta \equiv \log\delta$ and $\widehat{W}^\delta$, $\widehat{W}^{fi}$, and $\widehat{W}^{si}$ are independent Brownian motions under the probability measure of agent $i \in \{A, B\}$, as defined in Appendix A.1. Note that if $0 < \phi < 1$, agents come up with different estimates of λ because of the last term in (9).

This learning exercise results in particular dynamics of the filter. First, its instantaneous variance is directly driven by the fundamental. This is because agents are provided with more accurate information about the mean-reversion speed when $\bar{f} - f$ is large, as can be seen from (8). Consequently, the filter features stochastic volatility, which is higher when the fundamental is away from its long-term mean.

Second, the filter can exhibit regimes of positive or negative correlation with the fundamental. For example, if today the economy is in good times (i.e., $f_t > \bar{f}$) and agents observe

[10] Our setup can also be interpreted as a model of overconfidence (Scheinkman and Xiong, 2003): agents place infinite trust on their own signal and completely distrusts other information. An additional layer of heterogeneity can be added by assuming different parameters ϕ^A and ϕ^B (Alti and Tetlock, 2014). In this case, each agent perceives a distinct uncertainty about the future (typically, the uncertainty of the more over-confident agent will be lower). Although this would provide further insights into the interaction between disagreement and overconfidence, the addition of a state variable would unnecessarily complicate the exposition of our model. We further discuss in Section 4.2 comparative statics with respect to ϕ.

[11] See Theorem 12.7 in Liptser and Shiryaev (2001) and Appendix A.1 for details.

a positive change in the fundamental, then λ is likely to be small (i.e., the present boom is likely to persist). Following the same intuition, if the economy is in bad times (i.e., $f_t < \bar{f}$) and agents observe a positive change, then λ is likely to be large (i.e., the present recession is likely to be short). Agents thus form *extrapolative expectations*: they regard unusual good or bad past performance of the economy as indicators of a slow-moving economy, or as the economy's "new normal."[12]

Both the above-mentioned properties of the filter—stochastic volatility and asymmetric correlation—are endogenously generated from learning. A general critique of learning-based models is that filtered state variables have very similar properties with their unobservable counterparts, and thus incomplete information implies just a rewriting of the system of state variables, without much change in its properties. For instance, when agents learn about the level of an unobservable fundamental (e.g. Scheinkman and Xiong, 2003; Dumas et al., 2009; Whelan, 2014; Dumas et al., 2016), the fundamental and the filtered fundamental mean-revert at the same speed to the same long-term mean and both have constant volatility. This critique does not apply to our setting: while the unobservable variable λ has constant volatility and moves independently from other state variables, its estimate $\widehat{\lambda}$ features a U-shaped stochastic volatility and an asymmetric correlation with the fundamental. Both these features, which result endogenously from learning, will generate most of our results.

The Bayesian uncertainty γ follows the same dynamics for both agents

$$\frac{d\gamma_t}{dt} = (1-\phi^2)\Phi^2 - 2\kappa\gamma_t - \frac{(\bar{f}-f_t)^2}{\sigma_f^2(1-\rho^2)}\gamma_t^2. \tag{10}$$

When $f_t - \bar{f} \neq 0$, fluctuations in the fundamental are informative about λ. This effect is brought by the last term in (10), which causes uncertainty to constantly change and never converge to a constant. Learning about the persistence of the fundamental thus endogenously generates fluctuating uncertainty, which makes it different from learning about the fundamental: in the latter case, uncertainty converges rapidly to a constant steady state (e.g., Dumas et al., 2009).[13]

[12]This extrapolative nature of learning is different from "extrapolation bias," which refers to the tendency to overweight recent events when making decisions about the future (Hirshleifer et al., 2015). It is also different from "over-extrapolation," which refers to the tendency to believe that a stochastic process is more persistent than it actually is (Alti and Tetlock, 2014). In our case, both agents apply standard Bayesian rules (Brennan, 1998) and do not overweight recent events or wrongly perceive a more persistent fundamental process.

[13]Endogenously fluctuating uncertainty arises in other situations analyzed in the literature. As shown in Veronesi (2000), when drift of fundamentals shifts between two unobservable states at random times, uncertainty changes over time, being at its maximum when the probability of each of the two states is 0.5. This generates a hump-shaped uncertainty. In Xia (2001), learning is a function of the current state variable and thus uncertainty endogenously fluctuates over time.

The observable fundamental provides the link between the probability measures of the two agents, $\mathbb{P}^A$ and $\mathbb{P}^B$:

$$d\widehat{W}_t^{fA} = d\widehat{W}_t^{fB} + \frac{\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right)\left(\bar{f} - f_t\right)}{\sigma_f\sqrt{1-\rho^2}}dt, \tag{11}$$

where $\widehat{\lambda}^B - \widehat{\lambda}^A$ is the difference in beliefs about persistence. Since each agent perceives the economy under a different probability measure, the $\mathbb{P}^A$-expectation of any random variable X can also be computed under $\mathbb{P}^B$ by using the following relation

$$\mathbb{E}^A\left[X\right] = \mathbb{E}^B\left[\eta X\right], \tag{12}$$

where the process η is the change of measure from $\mathbb{P}^B$ to $\mathbb{P}^A$. Defining $\mathscr{O}_t$ as the observation filtration at time t, the change of measure η satisfies

$$\eta_t \equiv \left.\frac{d\mathbb{P}^A}{d\mathbb{P}^B}\right|_{\mathscr{O}_t} = e^{-\frac{1}{2}\int_0^t\left(\frac{\left(\widehat{\lambda}_t^B-\widehat{\lambda}_t^A\right)(\bar{f}-f_t)}{\sigma_f\sqrt{1-\rho^2}}\right)^2 ds - \int_0^t \frac{\left(\widehat{\lambda}_t^B-\widehat{\lambda}_t^A\right)(\bar{f}-f_t)}{\sigma_f\sqrt{1-\rho^2}} d\widehat{W}_s^{fB}}, \tag{13}$$

and thus has the following dynamics

$$\frac{d\eta_t}{\eta_t} = -\frac{\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right)\left(\bar{f} - f_t\right)}{\sigma_f\sqrt{1-\rho^2}}d\widehat{W}_t^{fB}. \tag{14}$$

We recover the customary change of measure obtained in the learning/difference-of-beliefs literature (e.g. Scheinkman and Xiong, 2003; Dumas et al., 2009; Whelan, 2014; Dumas et al., 2016), with an important distinction. In our setup, it is the interaction between the demeaned fundamental and the difference between agents' filtered mean-reversion speeds that drives the Radon-Nikodym derivative η. This interaction is absent in previous models, where η is solely driven by the difference between agents' filtered fundamentals.

Based on all the above considerations, we define two state variables particular to our setup of learning and difference-of-beliefs.

Definition 1. *The* ***structural disagreement*** *and the* ***structural uncertainty*** *in this economy are defined as*

$$\mathcal{D} \equiv (\bar{f} - f)(\widehat{\lambda}^B - \widehat{\lambda}^A) \tag{15}$$

$$\mathcal{U} \equiv (\bar{f} - f)\gamma. \tag{16}$$

These two state variables isolate the effects of disagreement and learning about the unobservable mean-reversion speed. Intuitively, the structural disagreement $\mathcal{D}$ only matters when the fundamental is away from its long-term mean; it is only then that agents' views of the economy differ through the change of measure η. Similarly, the structural uncertainty $\mathcal{U}$ only matters when the fundamental is away from its long-term mean. To gain some intuition on why this is the case, consider a simple Euler discretization of the fundamental process:

$$f_{t+1} = f_t + \lambda_t(\bar{f} - f_t) + \varepsilon_{t+1}. \tag{17}$$

Because λ_t is unobservable, the one-step ahead forecast error is given by

$$f_{t+1} - \mathbb{E}_t[f_{t+1}] = \varepsilon_{t+1} + (\lambda_t - \widehat{\lambda}_t)(\bar{f} - f_t), \tag{18}$$

which has two terms: the future random shock and the error arising from parameter uncertainty. The magnitude of this latter term grows with the distance between the fundamental and its long-term mean. In other words, uncertainty about the speed of mean-reversion λ matters only when the growth rate of the economy is away from its long-term mean, offering an intuitive justification of our notion of structural uncertainty.

To summarize, several features distinguish our incomplete information setup from previous work. First, the volatility of the filter is stochastic and increases when the fundamental is away from its long-term mean. Second, the sign of the correlation between the filter and the fundamental depends on economic conditions (negative in good times and positive in bad times). Third, uncertainty endogenously fluctuates over time and never converges to a constant. Finally, our model of learning generates two key quantities, structural disagreement and structural uncertainty, that are highly dependent on the state of the economy given by the value of the fundamental. As we describe below, all these features generate novel, testable implications for asset prices.

2.2 Equilibrium asset prices

For the rest of the paper, we choose to work under agent B's probability measure $\mathbb{P}^B$, without loss of generality. Assuming that markets are complete, we follow Cox and Huang (1989) and solve for the equilibrium using the martingale approach.[14] The state-price density perceived

[14]Following Dumas et al. (2009) and Dumas et al. (2016), we add to the existing stock and the locally risk-free asset a perpetual bond and two futures contracts whose underlyings are the information signals. That is, futures contracts are used to hedge fluctuations in information signals. Although our entire analysis abstracts from these three additional securities, they are needed to complete markets and therefore allow us to follow the martingale approach.

by agent B, ξ^B, satisfies

$$\xi_t^B = e^{-\beta t}\delta_t^{-\alpha}\left[\left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha} + \left(\frac{1}{\Lambda_B}\right)^{1/\alpha}\right]^{\alpha} \tag{19}$$

$$= \frac{1}{\Lambda_B}e^{-\beta t}\delta_t^{-\alpha}\omega_{Bt}^{-\alpha}, \tag{20}$$

where Λ_A and Λ_B are the Lagrange multipliers associated with the static budget constraints of agents A and B, and $\omega_{it} \equiv c_{it}/\delta_t$ is the consumption share of agent $i \in \{A, B\}$ at time t.[15] The consumption shares are functions of the change of measure η_t and their sum equals one.

According to (20), agent B's state-price density depends on the aggregate consumption δ but also on her consumption share ω_{Bt}. That is, the agent cares not only about the aggregate level of consumption (which would be the case in a representative agent economy), but also on how much of the aggregate output she shares with agent A.

Proposition 1. *The risk-free rate and the market price of risk perceived by agent B are:*

$$r_t = \beta + \alpha f_t - \frac{1}{2}\alpha(\alpha+1)\sigma_\delta^2 + \frac{1}{2}\frac{\alpha-1}{\alpha}\frac{\mathcal{D}_t^2}{\sigma_f^2(1-\rho^2)}\omega_{At}\omega_{Bt} \tag{21}$$

$$\theta_t^B = \begin{pmatrix}\alpha\sigma_\delta & \frac{\omega_{At}}{\sigma_f\sqrt{1-\rho^2}}\mathcal{D}_t & 0 & 0\end{pmatrix}^\top. \tag{22}$$

Assuming that the coefficient of relative risk aversion α is an integer,[16] *the equilibrium price-dividend ratio of the risky asset satisfies*

$$\frac{S_t}{\delta_t} = \sum_{j=0}^{\alpha}\binom{\alpha}{j}\omega_{At}^j\omega_{Bt}^{\alpha-j}F_j(Z_t), \tag{23}$$

where

$$F_j(Z_t) \equiv \mathbb{E}_t^B\left[\int_t^\infty e^{-\beta(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha}du\right]. \tag{24}$$

The 4-dimensional vector of state variables, Z, is defined by $Z \equiv [f, \widehat{\lambda}^A, \widehat{\lambda}^B, \gamma]^\top$.

Proof. See Appendix A.2. □

[15] Derivations are provided in Appendix A.2.

[16] This assumption simplifies the calculus. To the best of our knowledge, it has been first pointed out by Yan (2008) and Dumas et al. (2009). If the coefficient of relative risk aversion is real, the computations can still be performed using Newton's generalized binomial theorem. Bhamra and Uppal (2013) offer a comprehensive analysis for all possible values of the risk aversion.

By inspection of (21) and (22), structural disagreement is a key driver of both the risk-free rate and the market price of risk. The first three components in (21) form the usual risk-free rate in a representative agent economy. First, the risk-free rate increases with the discount factor β. Second, the risk-free rate increases with the fundamental f (in this case, agents expect higher future consumption and hence lower future marginal utility; future payments due to saving have lower value, which decreases the demand for the risk-free asset and increases the equilibrium risk-free rate). Third, the risk-free rate decreases with the volatility of aggregate consumption σ_δ (the higher the volatility, the more agents demand risk-free payments and hence a lower risk-free rate is necessary to clear the market for borrowing and lending).

The last term in (21) comprises an additional effect on the risk-free rate due to disagreement. When the risk aversion coefficient is larger than one, both agents in this economy expect a larger consumption share in the future under their own probability measure (see Equation (70) in Appendix A.2), and thus lower future marginal utility. This effect arises because each agent believes that the other agent's model is partially inaccurate. Therefore, in presence of disagreement, both agents decide to save less today, which increases the equilibrium risk-free rate.

The market price of risk perceived by agent B in (22) potentially loads on four independent sources of risk: aggregate consumption risk $\widehat{W}^{\delta}$, fundamental risk $\widehat{W}^{fB}$, agent B's information risk $\widehat{W}^{sB}$, and agent A's information risk $\widehat{W}^{sA*}$. Only aggregate consumption risk and fundamental risk are priced. First, agent B requires a positive price for bearing the risk of fluctuations in aggregate consumption δ, as determined by the first element in the vector (*market price of aggregate consumption risk*). Second, agent B requires a price for bearing fundamental risk which affects her consumption through the consumption share process (*market price of fundamental risk*). This second term exists because agents disagree about the interpretation of fundamental shocks.

To characterize the market price of fundamental risk, it is first important to interpret the sign of the structural disagreement $\mathcal{D}_t$. Whenever $\mathcal{D}_t > 0$, agent B's model has a *more favorable* economic outlook. This arises when (i) the economy is going through good times ($\bar{f} - f_t < 0$) and agent B believes the fundamental to be more persistent than agent A (a longer economic boom), or (ii) when the economy is going through bad times and agent B believes the fundamental to be less persistent than agent A (a shorter recession).

Given this interpretation, when agent B has a more favorable outlook ($\mathcal{D}_t > 0$), positive shocks to the fundamental increase her consumption share. This induces a positive correlation between her consumption share and the fundamental, and thus agent B requires a positive premium to bear fundamental risk. Alternatively, when $\mathcal{D}_t < 0$, agent B's model

has a less favorable outlook and her consumption share is negatively correlated with fundamental risk. She is then willing to pay a price to bear fundamental risk. Lastly, if $\mathcal{D}_t = 0$, agents have the same forecasts, consumption shares to not fluctuate, and thus there is no market price of fundamental risk. Note also that agent B requires a larger market price of risk (in absolute value) when the proportion of agent A in the economy, ω_A, is large.

2.3 Stock-return volatility and equity risk premium

Proposition 2 below characterizes the equilibrium stock-return volatility and risk premium.

Proposition 2. *The state variables in this economy are f_t, $\widehat{\lambda}_t^A$, $\widehat{\lambda}_t^B$, γ_t, and $\mu_t \equiv \log \eta_t$. The diffusion vector of stock returns, Σ, and the risk premium perceived by agent B, RP_B, satisfy*

$$\Sigma = \begin{pmatrix} \sigma_\delta + \frac{S_f}{S}\sigma_f \rho \\ \frac{S_f}{S}\sigma_f\sqrt{1-\rho^2} + \frac{1}{\sigma_f\sqrt{1-\rho^2}}\left[-\frac{S_\mu}{S}\mathcal{D} + \left(\frac{S_{\lambda_A}}{S} + \frac{S_{\lambda_B}}{S}\right)\mathcal{U}\right] \\ \phi\Phi\left(\frac{S_{\lambda_A}}{S}\phi^2 + \frac{S_{\lambda_B}}{S}\right) \\ \frac{S_{\lambda_A}}{S}\phi\Phi\sqrt{1-\phi^4} \end{pmatrix}^\top \tag{25}$$

$$RP_B \equiv \Sigma\theta^B \tag{26}$$

$$= \alpha\sigma_\delta\left(\sigma_\delta + \frac{S_f}{S}\sigma_f\rho\right) + \frac{S_f}{S}\mathcal{D}\omega_A + \left[\left(\frac{S_{\lambda_A}}{S} + \frac{S_{\lambda_B}}{S}\right)\mathcal{U}\mathcal{D} - \frac{S_\mu}{S}\mathcal{D}^2\right]\frac{\omega_A}{\sigma_f^2(1-\rho^2)} \tag{27}$$

where S_y denotes the partial derivative of the stock price with respect to the state variable y, and $\mathcal{D}$ and $\mathcal{U}$ are defined in (15) and (16).

Proof. Application of Itô's lemma on the stock price defined in (23). □

Because the economy is driven by four sources of risk, the stock-return diffusion Σ, expressed in Equation (25), has four components. The instantaneous volatility of stock returns is then computed as the norm of the diffusion vector, $\sigma \equiv ||\Sigma||$. It is important to notice that the structural disagreement $\mathcal{D}$ and the structural uncertainty $\mathcal{U}$ directly impact the stock-return volatility. We analyze the impact of these objects in Section 3.2.

The risk premium perceived by agent B in (27) is computed as the vector product between the market price of risk θ^B and the stock return diffusion Σ. Because the market price of risk is driven solely by structural disagreement, it follows that disagreement directly impacts the risk premium in the economy. Furthermore, Equation (27) shows that if we set $\mathcal{D} = 0$, then uncertainty has no direct impact on the risk premium. This implies that disagreement is an important channel through which uncertainty impacts the risk premium in our model. We will explore these implications in Section 3.3.

Learning about the length of business cycles connects both uncertainty and disagreement with the observable fundamental f, as shown in Definition 1. This feature is particular to our model and implies that uncertainty and disagreement have an impact only when the fundamental is away from its long-term mean (i.e., when $f \neq \bar{f}$). As the fundamental is closer to its long-term mean, both structural uncertainty and structural disagreement get close to zero. Conversely, a fundamental far from its long-run mean enhances the effects of uncertainty and disagreement on asset returns. We turn now to a quantitative examination of these effects.

3 Empirical Evidence

We begin by calibrating the parameters that the agents observe to real U.S. output growth data. Then, using those results, we provide numerical analysis to charaterize how uncertainty and disagreement affect asset prices. Finally, we develop a set of precise empirical predictions and test them using quarterly S&P 500 data.

3.1 Calibration

Set $\alpha = 3$, $\beta = 0.03$, and the ratio of Lagrange multipliers equal to one.[17] We calibrate the model to the following data: (i) the real U.S. GDP growth rate; (ii) the median of analyst forecast of the U.S. GDP growth rate for the current quarter; (iii) the 75th percentile of the 1-quarter-ahead analyst forecasts; and (iv) the 25th percentile of the 1-quarter-ahead analyst forecasts. We calibrate all parameters using Maximum Likelihood estimation. The data is at quarterly frequency, from Q4:1968 to Q2:2016, and obtained from the Federal Reserve Bank of Philadelphia.[18]

The parameter estimates and their statistical significance are summarized in Table 1.[19] All parameters are statistically significant at the 99% confidence level. The volatility of dividend growth σ_δ, the long-term mean of the fundamental $\bar{f}$, and the volatility of the fundamental σ_f are consistent with the values used in the asset-pricing literature (e.g., Brennan and Xia, 2001; Bansal and Yaron, 2004; Bansal, Kiku, and Yaron, 2010; Croce, Lettau, and Ludvigson, 2014). The correlation ρ between the fundamental and the dividend growth rate is positive (analysts adjust their forecasts upward when they observe a

[17] This choice of risk aversion and the subjective discount rate yields a mean price-dividend ratio of about 17 and a mean stock-return volatility of about 16%. Equal Lagrange multipliers ensures that agents are endowed with the same initial share of consumption

[18] https://www.philadelphiafed.org/research-and-data/

[19] See Appendix A.4 for further details on the estimation method.

Parameter	Symbol	Estimate
Volatility of dividend growth	σ_δ	0.0124^{***} (8.00×10^{-4})
Long-term fundamental	$\bar{f}$	0.0220^{***} (1.00×10^{-4})
Volatility of fundamental	σ_f	0.0233^{***} (4×10^{-4})
Correlation between dividend and fundamental	ρ	0.0694^{***} (1.95×10^{-2})
Mean-reversion speed of fundamental mean-reversion speed	κ	0.2014^{***} (2.97×10^{-2})
Long-term fundamental mean-reversion speed	$\bar{\lambda}$	0.2165^{***} (6.7×10^{-3})
Volatility of fundamental mean-reversion speed	Φ	0.0435^{***} (5.4×10^{-3})
Difference-of-beliefs parameter	ϕ	0.3754^{***} (4.07×10^{-2})

Table 1: Calibration to the U.S. economy (Maximum Likelihood estimation) Parameter values resulting from a Maximum Likelihood estimation, as described in Appendix A.4. Standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is respectively labeled *, **, and ***.

positive growth surprise, and downward when they observe a negative growth surprise), but lower than one (analysts use other sources of information to infer the expected growth). The long-term mean-reversion speed of the fundamental implies a half-life of approximately three years, which is slightly shorter than what is assumed in the long-run risk literature. The mean-reversion speed itself is relatively persistent as its half-life is approximately four years. Furthermore, its volatility is significantly different from zero, which lends support to our assumption of time-varying mean-reversion speed.

3.2 Stock-return volatility

We turn now to characterizing how uncertainty and disagreement affect stock return volatility using the calibration in Table 1. We solve numerically for equilibrium quantities using the Chebyshev collocation method (details are provided in Appendix A.3). We then compute the stock return volatility by means of Equation (25) of Proposition 2. In Figure 1, we plot the volatility against the deviation between the long-term mean of the fundamental and

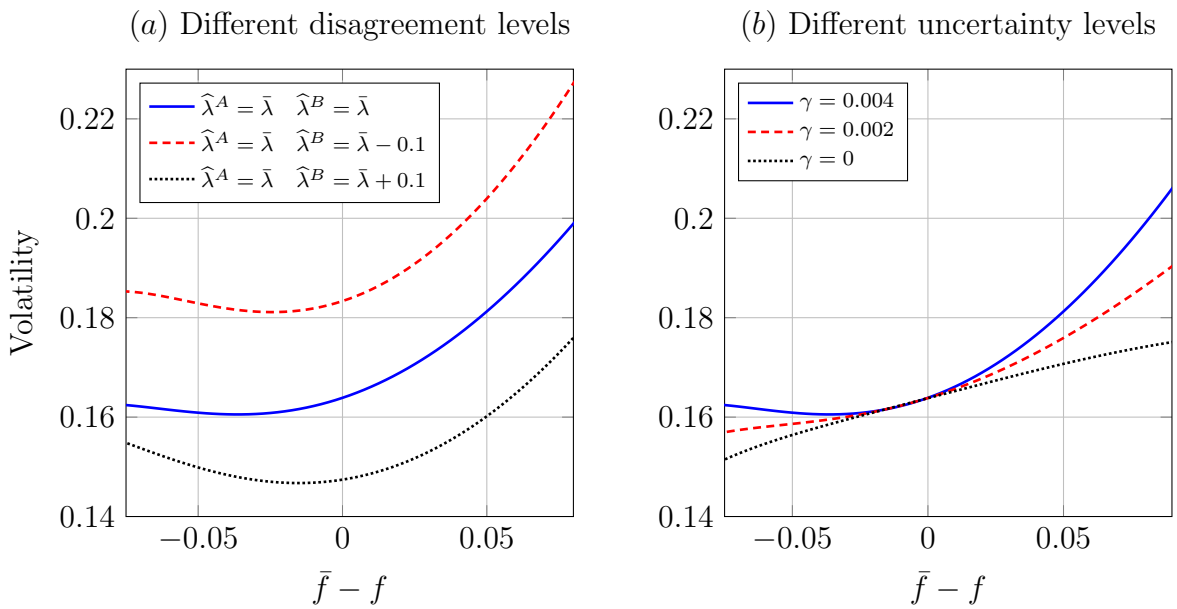

Figure 1: Volatility vs. $(\bar{f} - f)$ and uncertainty γ

Panel (a) plots the stock-return volatility against $(\bar{f} - f)$ for three different values of disagreement. For this panel, the Bayesian uncertainty γ is fixed at $\bar{\gamma} = 0.004$ (as defined in Appendix A.1). Panel (b) plots the stock-return volatility against $(\bar{f} - f)$ for three different values of the uncertainty γ. For this panel, disagreement is fixed at $\Delta\widehat{\lambda} \equiv \widehat{\lambda}^B - \widehat{\lambda}^A = 0$. Unless otherwise specified, parameter values are provided in Table 1.

the fundamental, i.e. $\bar{f} - f$. Note that this term appears in the definition of structural uncertainty and structural disagreement, which justifies its choice for the plot. As such, in Figure 1, the right-hand side of zero can be interpreted as "bad times," whereas the left-hand side can be interpreted as "good times."

Panel (a) plots the stock return volatility for three different levels of disagreement. Here, we fix $\widehat{\lambda}^A = \bar{\lambda}$, with $\bar{\lambda}$ defined in Table 1, and assume that $\widehat{\lambda}^B$ takes three different values such that $\Delta\widehat{\lambda} = \widehat{\lambda}^B - \widehat{\lambda}^A \in \{0, -0.1, 0.1\}$. Panel (b) plots the stock return volatility for three different levels of uncertainty, $\gamma \in \{\bar{\gamma}, \bar{\gamma}/2, 0\}$, where $\bar{\gamma}$ represents an upper bound which arises from learning (we define and interpret this upper bound in Appendix A.1).

Both panels of Figure 1 show that stock return volatility significantly varies with the fundamental. Volatility is U-shaped in $(\bar{f} - f)$ and reaches a minimum when the fundamental is close to its long-term mean. The feature that generates this U-shaped pattern is the stochastic volatility of the filter, which we describe in Section 2.1. More precisely, when the fundamental is away from its long-term mean, agents accurately update their assessment of λ using fundamental shocks; this amplifies the effect of fundamental shocks, generating higher volatility of the filter and hence higher stock-return volatility.

Focusing on panel (a), disagreement has an additional impact on the level of volatility,

beyond the impact of the fundamental. Keeping $\widehat{\lambda}^A$ fixed at $\bar{\lambda}$, volatility increases when one of the agents—in this particular case agent B—perceives more persistence in the expected growth rate (dashed line). The three lines in the plot show that this effect is non-negligible: when one agent believes the economy to be more persistent, fundamental shocks have a long-lasting impact on the future growth, thus generating higher stock-return volatility.

Turning now to panel (b), most of the intuition can be conveyed by starting with the case when uncertainty is zero (dotted line). In this case, volatility increases almost linearly with $(\bar{f} - f)$ and is thus higher in bad times (more on this asymmetric effect below). Then, uncertainty amplifies the effect, but only when the fundamental is away from its mean, generating a "hockey-stick" pattern.

The asymmetric effect in panel (b), i.e., higher volatility in bad times, results from the switching sign of the correlation between the filter and the fundamental, which we describe in Section 2.1. More precisely, the filter and the fundamental are positively correlated in bad times and negatively correlated in good times. Shocks in bad times are therefore amplified. For instance, on top of a bad fundamental shock occurring in bad times, agents update and believe the economy is more persistent. Both of these shocks move prices in the same direction and thus magnify the volatility in bad times. In contrast, in good times, when agents update, this moves the filter in the opposite direction as the fundamental, which dampens the volatility of stock returns.

To summarize, changes in the fundamental have a first-order effect on the volatility of asset returns, which increases in bad times. Adding to this effect, both the uncertainty and the disagreement about the mean-revering speed amplify fundamental shocks when the economy is away from its long-term mean. These interactions generate an asymmetric U-shaped (hockey-stick) volatility pattern. As we will show in Section 3.4.1, these interactions also suggest an approach to empirically test our novel theoretical predictions.

3.3 Equity risk premium

According to our discussion of the market price of fundamental risk in Section 2.2, $\mathcal{D} > 0$ implies that agent B has a more favorable economic outlook. With this interpretation in mind, we plot the equity risk-premium perceived by agent B (Equation (27) of Proposition 2) as a function of the structural disagreement $\mathcal{D}$ in the two panels of Figure 2. Panel (a) depicts this relationship for two different levels of disagreement. Here, we assume that $\widehat{\lambda}^A = \bar{\lambda}$, we fix $\widehat{\lambda}^B$ at two different values such that $\Delta\widehat{\lambda} \in \{-0.1, 0.1\}$, and then vary the level of the fundamental in order to obtain different values of structural disagreement $\mathcal{D}$.

Panel (a) shows that the risk premium required by agent B for holding the risky asset is

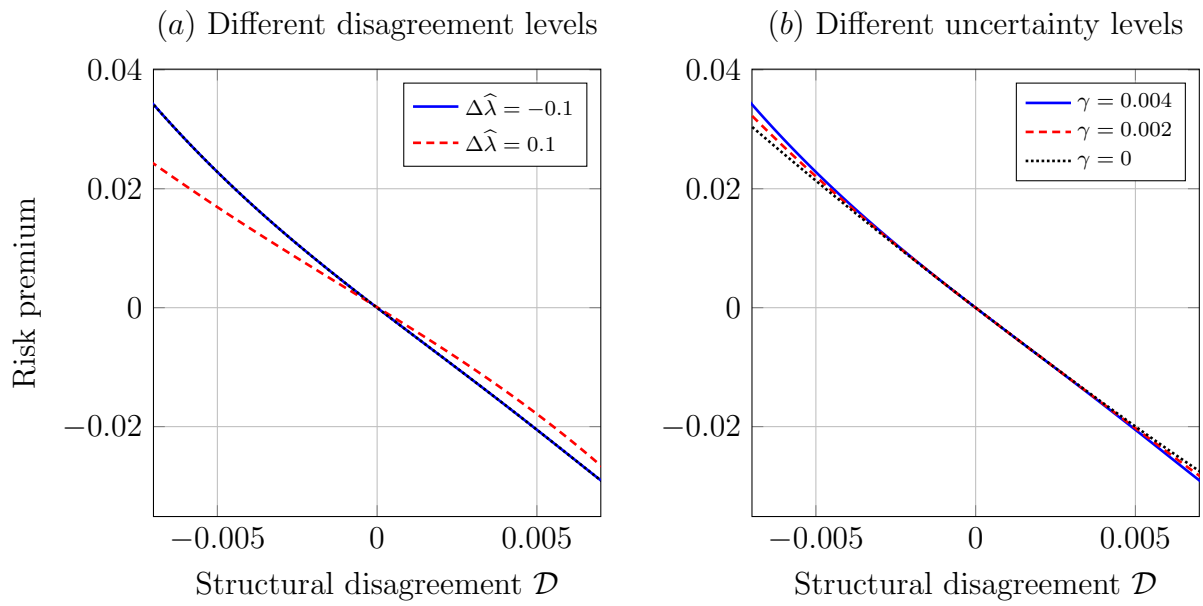

Figure 2: Risk premium vs. structural disagreement $\mathcal{D}$

Panel (a) plots the equity risk premium perceived by agent B against the structural disagreement $\mathcal{D} = (\bar{f} - f)\Delta\widehat{\lambda}$ for two different values of disagreement. For this panel, the Bayesian uncertainty γ is fixed at $\bar{\gamma} = 0.004$ (as defined in Appendix A.1). We then vary the fundamental f by keeping $\Delta\widehat{\lambda}$ constant in order to obtain different levels of structural disagreement. Panel (b) plots the equity risk premium for three different values of the Bayesian uncertainty γ. For this panel, disagreement is fixed to $\Delta\widehat{\lambda} = -0.1$ (i.e., agent B believes the fundamental is more persistent). Parameter values are provided in Table 1.

high when her model forecasts a less favorable economic outlook, i.e., when $\mathcal{D} < 0$ (notice that this result is independent on the sign of $\Delta\widehat{\lambda}$). When $\mathcal{D} < 0$, the stock is a bad hedge against fundamental risk—fluctuations in the stock price are positively related to fluctuations in agent B's consumption share—and thus agent B requires a large risk premium to hold the risky asset. Conversely, when agent's B economic outlook is more favorable ($\mathcal{D} > 0$), fluctuations in the stock price are negatively related to fluctuations in her consumption share, which makes the stock a good hedge against fundamental risk and therefore she is willing to pay a premium to hold the risky asset.

Panel (b) of Figure 2 plots the risk premium for three different levels of uncertainty, $\gamma \in \{\bar{\gamma}, \bar{\gamma}/2, 0\}$, with disagreement now fixed at $\Delta\widehat{\lambda} = -0.1$. As Proposition 1 suggests, structural uncertainty affects the risk premium only when disagreement is present. Nonetheless, the plot shows that this effect is not quantitatively strong, at least with this calibration.

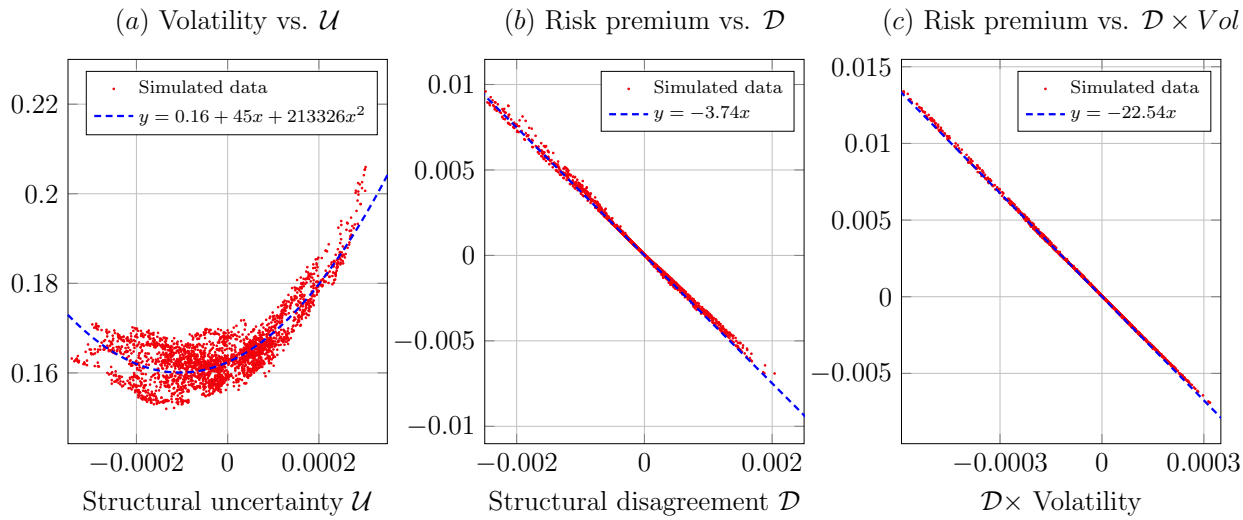

Figure 3: Three testable theoretical predictions (model-simulated data)
Panel (a) plots the stock-return volatility against the structural uncertainty $\mathcal{U}$ with model-generated data resulting from one simulation of the economy at weekly frequency over 50 years (2,600 data points). The dashed line represents a quadratic fit; the equation of the regression line is shown in the legend. Panel (b) plots the risk premium against the structural disagreement $\mathcal{D}$ with the same model-generated data. The dashed line represents the linear fit; the equation of the regression line is shown in the legend. Panel (c) plots the risk premium against the product $\mathcal{D} \times Vol$. The dashed line represents the linear fit; the equation of the regression line is shown in the legend. Parameter values are provided in Table 1. Initial values for the simulation are $f_0 = \bar{f}$, $\widehat{\lambda}_0^A = \widehat{\lambda}_0^B = \bar{\lambda}$, $\delta_0 = \eta_0 = 1$, and $\gamma_0 = \bar{\gamma}$, where $\bar{\gamma}$ is defined in Appendix A.1.

3.4 Empirical Tests

3.4.1 Predictions

To characterize the predictions of our model, we perform simulations of the economy over 50 years at weekly frequency and compute for each simulated point four endogenous quantities: stock return volatility, equity risk premium, structural uncertainty, and structural disagreement.

The first testable prediction is that increased structural uncertainty is associated with a non-linear increase in volatility (Figure 1). We highlight this relationship in panel (a) of Figure 3, where we plot the volatility of stock returns against the structural uncertainty $\mathcal{U}$. We add a quadratic fit line (equation given in the legend of the plot), which confirms the asymmetric hockey-stick pattern.

The second testable prediction is that the equity risk premium perceived by agent B decreases with structural disagreement (Figure 2). Panel (b) plots the risk premium perceived by agent B against the structural disagreement $\mathcal{D}$ and confirms the decreasing relationship,

with the equation of the linear fit shown in the legend of the plot.

Finally, the third theoretical prediction concerns the relation between volatility and the risk premium (i.e., the risk-return tradeoff). Our model predicts that this relation is ambiguous, which might rationalize the inconclusive findings in the literature about the sign and the significance of this relationship.[20] More precisely, the risk premium given in Equation (27) of Proposition 2 can be further written:

$$RP_B = a_0 + a_1 \times f(\mathcal{U}, \mathcal{D})\mathcal{D}, \tag{28}$$

where $f(\mathcal{U}, \mathcal{D})$ is a function directly related to the volatility of stock returns and a_0 and a_1 are two coefficients that do not depend on $\mathcal{U}$ and $\mathcal{D}$. Equation (28) suggests a linear relationship between risk premium and the *volatility interacted with structural disagreement*. Because structural disagreement can be either positive or negative, the risk-return tradeoff becomes ambiguous. Yet, panel (c) shows that in our model the risk premium perceived by agent B is clearly higher when the product between structural disagreement and volatility is large and negative. In other words, conditional on structural disagreement being negative, there exists a positive risk-return tradeoff in this economy. Conditional on structural disagreement being positive, however, the risk-return tradeoff is negative. This is our third testable theoretical prediction.

3.4.2 Results

We use quarterly S&P 500 excess returns from Q4:1968 to Q2:2016[21] to build empirical proxies for the risk premium (which we denote by RP^e) and for the volatility (which we denote by Vol^e). For the risk premium, we model excess returns as an $AR(p)$ process, where p is the number of lags. We set the number of lags to $p = 1$ (only the first lagged return of the estimated $AR(p)$ features a significant loading). The risk premium is then defined as the fitted value of the estimated $AR(1)$ process.[22] We then assume that the residuals of the $AR(1)$ process follow a $GARCH(1,1)$ model, which yields the stock return volatility.[23]

Our empirical strategy requires proxies for the fundamental f^e, the disagreement $\Delta\widehat{\lambda}^e \equiv \widehat{\lambda}^{B,e} - \widehat{\lambda}^{A,e}$, and the uncertainty γ^e. We build these proxies from the Maximum Likelihood

[20] See Glosten et al. (1993), Brandt and Kang (2004), Rossi and Timmermann (2015) and the references therein.

[21] http://www.econ.yale.edu/ shiller/data.htm

[22] Assuming an $ARMA(p,q)$ instead of an $AR(p)$ does not change our results. Moreover, adding the dividend yield as a predictor of future returns (Xia, 2001; Cochrane, 2008; Van Binsbergen and Koijen, 2010) does not affect our results either.

[23] The results hold if we use either a $GARCH$ model with additional lags or an asymmetric $GARCH$ model to proxy for stock return volatility.

estimation performed in Section 3.1. Specifically, with the calibration of Table 1 and the four time series used in the estimation, we build model-implied time series of the four Brownian motions governing the economy (more details in Appendix A.4). With these time series at hand, we construct empirical proxies for f^e, $\Delta\widehat{\lambda}^e$, and γ^e. The empirical proxies for structural uncertainty and structural disagreement are then defined as $\mathcal{U}^e \equiv (\bar{f} - f^e)\gamma^e$ and $\mathcal{D}^e \equiv (\bar{f} - f^e)\Delta\widehat{\lambda}^e$, respectively.

Our first empirical test is based on the following regression:

$$Vol_t^e = a_0 + a_1\mathcal{U}_t^e + a_2(\mathcal{U}_t^e)^2 + \varepsilon_t, \tag{29}$$

Column (a) of Table 2 presents the results, lending support to our first prediction: the S&P 500 return volatility tends to increase with structural uncertainty; moreover, the S&P 500 return volatility is a quadratic function of structural uncertainty.

One potential concern is that the coefficients in column (a) are significant not because the interaction per se is a driver of volatility but instead because one of the two components of structural uncertainty significantly drives the dependent variable. Therefore we consider column (b) and control for the individual components of the structural uncertainty. We can see that both the linear and quadratic coefficients a_1 and a_2 remain positive and significant. Furthermore, fixing $\bar{f} - f^e = 0$ yields an insignificant loading on γ^e, which supports our prediction that uncertainty about the length of business cycles matters only when the fundamental is away from its long-term mean. We depict the quadratic relation in panel (a) of Figure 4 (akin with panel (a) of Figure 3). Indeed, there exist a positive relation between volatility and structural uncertainty in the data, and the relation features a hockey-stick pattern, as our model predicts.

Our second empirical test is based on the following regression:[24]

$$RP_t^e = a_0 + a_1\mathcal{D}_t^e + \varepsilon_t, \tag{30}$$

Column (c) of Table 2 presents the results. The negative and significant coefficient a_1 is consistent with our model's prediction that the S&P 500 risk premium represents the view of the pessimistic agent and decreases with structural disagreement. This negative relation is highly statistically significant, even when the two individual components of structural

[24] An important question for the interpretation of this test is which agent's risk premium do we measure empirically—is it the one with the optimistic economic outlook or the one with the pessimistic economic outlook? In our calibration on analyst forecast data (Section 3.1), we assume that agent B's forecast of the one-quarter ahead growth rate always represents the 25th percentile. Accordingly, agent B is calibrated to be on average the one with the less favorable economic outlook. A negative (and significant) coefficient a_1 in the regression (30) would then confirm that we measure the risk premium perceived by agent B.

	(a) Vol^e	(b) Vol^e	(c) RP^e	(d) RP^e	(e) RP^e	(f) RP^e
Const.	0.148***	1.08	0.0068***	0.0069***	0.0070***	0.008***
	(0.006)	(0.716)	(0.0007)	(0.0034)	(0.0007)	(0.003)
$\mathcal{U}^e$	144.9***	9993*				
	(53.4)	(5802)				
$(\mathcal{U}^e)^2$	730591*	525328*				
	(4.33×10^5)	(2.78×10^5)				
$\mathcal{D}^e$			−1.719***	−1.874***		
			(0.736)	(0.588)		
$\mathcal{D}^e \times Vol^e$					−14.22***	−13.33***
					(5.08)	(4.16)
$\bar{f} - f^e$		−39.1*		−0.142***		−0.126***
		(23.1)		(0.043)		(0.044)
γ^e		−232.5				
		(178.7)				
$\Delta\widehat{\lambda}^e$				−0.006		−0.004
				(0.009)		(0.008)
Vol^e						−0.008
						(0.023)
R^2	0.166	0.203	0.051	0.149	0.105	0.190
# Obs.	191	191	191	191	191	191

Table 2: Empirical tests of the relations between S&P 500 return volatility and risk premium vs. structural uncertainty and disagreement
Columns (a) and (b) present results of the model (29), with and without controls. Columns (c) and (d) present results of the model (30), with and without controls. Columns (e) and (f) present results of the model (31), with and without controls. Newey and West (1987) standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is respectively labeled *, **, and ***.

disagreement are considered as control variables. Furthermore, fixing $\bar{f} - f^e = 0$ yields an insignificant loading on $\Delta\widehat{\lambda}^e$, which supports our prediction that disagreement about the length of business cycles matters only when the fundamental is away form its long-term mean. We depict this relation in panel (b) of Figure 4 (akin with panel (b) of Figure 3).

Our third and final empirical test is based on the following regression:

$$RP_t^e = a_0 + a_1(\mathcal{D}_t^e \times Vol_t^e) + \varepsilon_t. \tag{31}$$

Columns (e) and (f) of Table 2 present the results, which are consistent with our model's prediction. The risk-return tradeoff becomes apparent when the volatility is interacted with

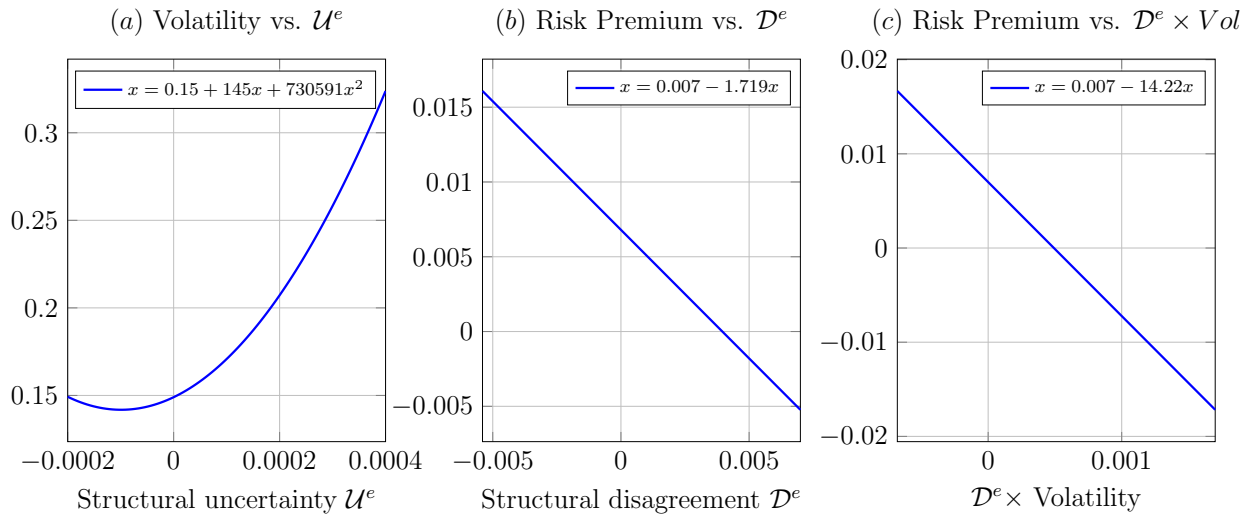

Figure 4: Empirical relationships between volatility, risk premium, structural uncertainty, and structural disagreement
Panel (a) plots the quadratic fit resulting from the regression in column (a) of Table 2, with the regression specification given in (29). Panel (b) plots the quadratic fit resulting from the regression in column (c) of Table 2, with the regression specification given in (30). Panel (c) plots the quadratic fit resulting from the regression in column (e) of Table 2, with the regression specification given in (31).

the structural disagreement. The relationship is negative and strongly significant, even after controlling separately for volatility and the individual components of structural disagreement.[25] We depict this relation in panel (c) of Figure 4 (akin with panel (c) of Figure 3).

4 Additional Results and Robustness Checks

4.1 Term structure of disagreement

Define disagreement at horizon τ as the absolute difference between the agents' expectations of the fundamental at the τ-year horizon:

$$D(\tau) \equiv \left| \mathbb{E}^A\left(f_\tau\right) - \mathbb{E}^B\left(f_\tau\right) \right|, \tag{32}$$

where $\mathbb{E}^A(.)$ and $\mathbb{E}^B(.)$ are expectations taken under the probability measures of agents A and B. We compare this term structure of disagreement with its empirical counterpart, $D^e(\tau)$, which we define as the median difference between the 75- and 25-percentile of the τ-year-

[25] The results are similar if the vector of controls is defined as $[\mathcal{D}^e, Vol^e]$. In this case, the loading on Vol^e is also insignificant, yielding support to our prediction that volatility is priced only when disagreement is present.

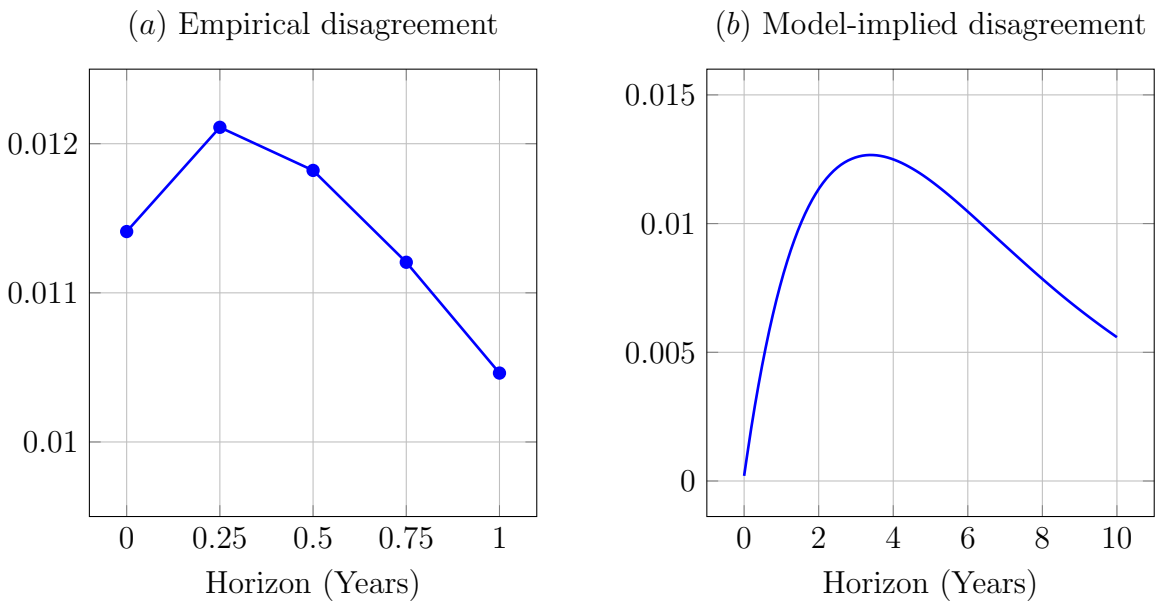

Figure 5: Empirical and model-implied term structures of disagreement
Panel (a) plots the empirical term structure of disagreement: disagreement at horizon τ, $D^e(\tau)$, is measured as the median of the difference between the 75th and 25th percentile of the τ-year-ahead analyst forecasts on real U.S. GDP growth (obtained from the Federal Reserve Bank of Philadelphia). Panel (b) plots the model-implied term structure of disagreement: disagreement at horizon τ, $D(\tau)$, is measured as the absolute difference between the two agents' expectations of the fundamental at the τ-year horizon. Parameter values are provided in Table 1. Initial values are $f_0 = 6\%$, $\widehat{\lambda}_0^A = \bar{\lambda} + 2\Phi/\sqrt{2\kappa}$, $\widehat{\lambda}_0^B = \bar{\lambda} - 2\Phi/\sqrt{2\kappa}$, $\delta_0 = \eta_0 = 1$, and $\gamma_0 = \bar{\gamma}$, with $\bar{\gamma}$ defined in Appendix A.1.

ahead analyst forecast on real U.S. GDP growth (computed over the number of observations), with τ varying from the current quarter to 1-year ahead.

We plot the empirical and model-implied term structures of disagreement in Figure 5. The empirical term structure of disagreement is slightly hump-shaped (left panel), being higher at the 1-quarter horizon and decreasing thereafter. The literature presents mixed conclusions about the shape of the term structure of disagreement: Patton and Timmermann (2010) provide evidence of an increasing term structure of disagreement, whereas Andrade et al. (2017) argue for a decreasing term structure. Our analysis suggests a non-monotonic pattern, which can be either increasing or decreasing depending on the time horizon considered.

Turning to the model-implied term structure of disagreement (right panel), we also obtain a hump-shaped pattern. This pattern is implied by our learning exercise: agents agree on the value of the fundamental today, but disagree over its future path, which generates the hump-shape. In contrast, models of dispersion of beliefs where agents disagree about the level of the fundamental (e.g. Dumas et al., 2009) can only generate a monotonically decreasing term structure of disagreement. The two plots appear to validate our motivation to study

agent disagreement about the persistence of the fundamental and not necessarily about its level (with the caveat that we do not exactly replicate the observed timing of disagreement, which peaks at the 1-quarter horizon in the data and at the 3-year horizon in our model).

4.2 Disagreement and overconfidence

The parameter ϕ affects how much disagreement is in the market and how much uncertainty agents face. Higher ϕ implies better information and less uncertainty. Because in our setting agents do not trust the information of others, this parameter may also be interpreted as "overconfidence" (Alti and Tetlock, 2014). We analyze now the role played by this parameter in our results.

In Appendix A.2, we show that the dynamics of disagreement are given by:

$$d\Delta\widehat{\lambda}_t = -\left(\kappa + \frac{\gamma_t\left(\bar{f} - f_t\right)^2}{\sigma_f^2(1-\rho^2)}\right)\Delta\widehat{\lambda}_t dt + \begin{pmatrix} 0 & 0 & \phi\Phi\left(1-\phi^2\right) & \phi\Phi\sqrt{1-\phi^4}\end{pmatrix} d\widehat{W}_t, \tag{33}$$

which implies that ϕ affects the instantaneous volatility of disagreement. As previously discussed in Section 2.1, when ϕ is either zero or one, the agents are in perfect agreement. The instantaneous volatility of disagreement is magnified in our model when ϕ takes an intermediate value (i.e., when $\phi \approx 0.7$).

In Figure 6, we consider two different values of ϕ (the calibrated value of Table 1 is $\phi = 0.3754$): $\phi = 0.1$ and $\phi = 0.7$. In the first case, fluctuations in disagreement are relatively small when compared to our benchmark calibration, whereas in the latter case, the fluctuations are larger. We then simulate the economy as in Section 3.4.1 and make the same plot as panel (a) of Figure 3. The two plots in Figure 6 show the role played by the parameter ϕ. When $\phi = 0.1$, structural uncertainty is the sole driver of volatility and fluctuations in disagreement do not add much variability. When $\phi = 0.7$, fluctuations in disagrement are more important and explain a lot of the variation in volatility. This suggests that overconfidence enhances disagreement in the market and generates stronger fluctuations in stock market volatility.

4.3 Uncertainty and the persistence of volatility

Fluctuations in structural uncertainty $\mathcal{U}$ affect fluctuations in volatility. This implies that the persistence of stock-return volatility is directly related to the persistence of uncertainty. Given that structural uncertainty is endogenously generated by the learning of agents, it

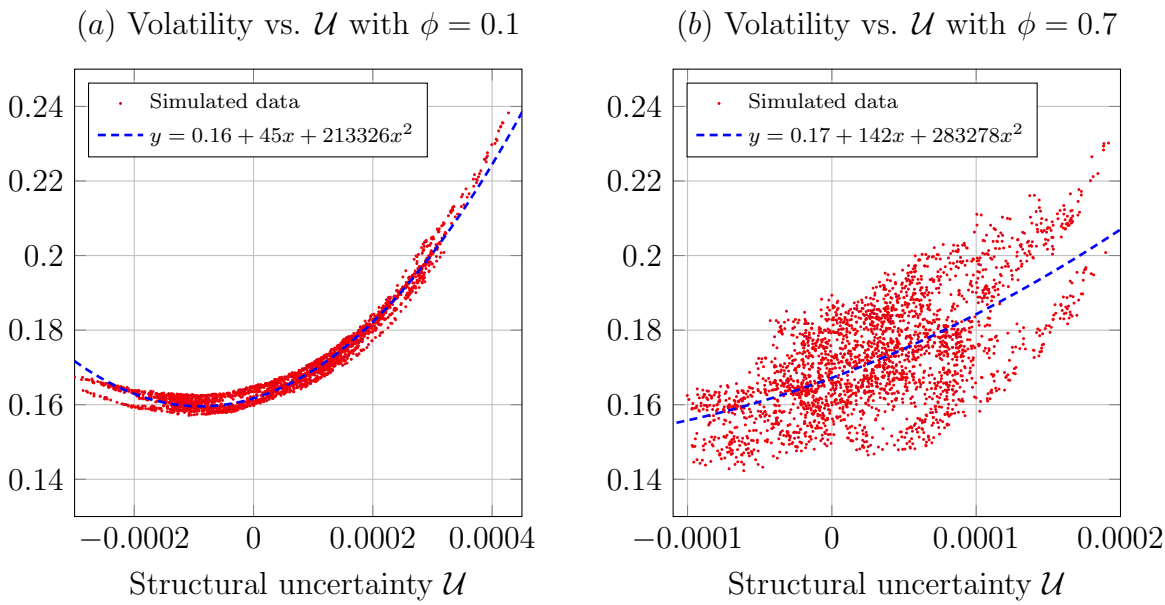

Figure 6: The role of the parameter ϕ

Both panels plot the stock-return volatility against the structural uncertainty $\mathcal{U}$ with model-generated data resulting from one simulation of the economy at weekly frequency over 50 years (2,600 data points). The dashed lines represent quadratic fits, with regression equations shown in the legends. In panel (a) we fix $\phi = 0.1$, whereas in panel (b) we fix $\phi = 0.7$. Unless otherwise specified, parameter values are provided in Table 1.

is a strongly persistent process.[26] Stock-return volatility is thus persistent not necessarily because the fundamental is assumed to be persistent, but primarily because the uncertainty resulting from learning about the mean-reversion speed λ is persistent. This offers a plausible theoretical foundation of the GARCH behavior commonly observed in financial markets (Engle, 1982; Bollerslev, 1986).

One way to show that persistence in uncertainty accounts for persistence in stock-return volatility is to consider different values for $\bar{\lambda}$. In Table 3, we estimate the 1-month autocorrelation of volatility (on model-simulated data) for different values of $\bar{\lambda}$. As $\bar{\lambda}$ increases, the fundamental becomes less autocorrelated but uncertainty remains strongly persistent, and so does the equilibrium stock-return volatility.

In our model, disagreement about the length of the business cycle is the main driver of

[26]It is straightforward to understand why learning implies persistent uncertainty. If uncertainty is high, agents need not one but a succession of high quality signals for uncertainty to decrease significantly (conversely, a succession of low quality signals are needed for uncertainty to increase significantly). Collin-Dufresne et al. (2015) show that, when investors need to learn about the constant expected consumption growth rate of the economy, uncertainty is strongly persistent and converges to a constant steady state. In their equilibrium model, such persistence implies a large risk premium when investors have Epstein and Zin (1989) preferences with a sufficiently large elasticity of intertemporal substitution. In our model, uncertainty is indeed persistent but never converges to a constant steady state because its dynamics depend on the fundamental, as shown in (10).

Parameter values	Persistence fundamental	Persistence uncertainty	Persistence volatility
Benchmark calibration	0.972	0.998	0.987
$\bar{\lambda} = 1$	0.923	0.997	0.986
$\bar{\lambda} = 2$	0.808	0.994	0.982
$\bar{\lambda} = 5$	0.632	0.990	0.981

Table 3: Persistence in stock-return volatility vs. persistence in fundamental
This table reports the 1-month autocorrelation in the fundamental f, uncertainty γ, and stock-return volatility. Unless stated otherwise, parameter values are provided in Table 1. In rows 2 to 4, the parameter σ_f is chosen such that the long-term volatility of the fundamental, $\sigma_f/\sqrt{2\bar{\lambda}}$, is equal to its benchmark counterpart.

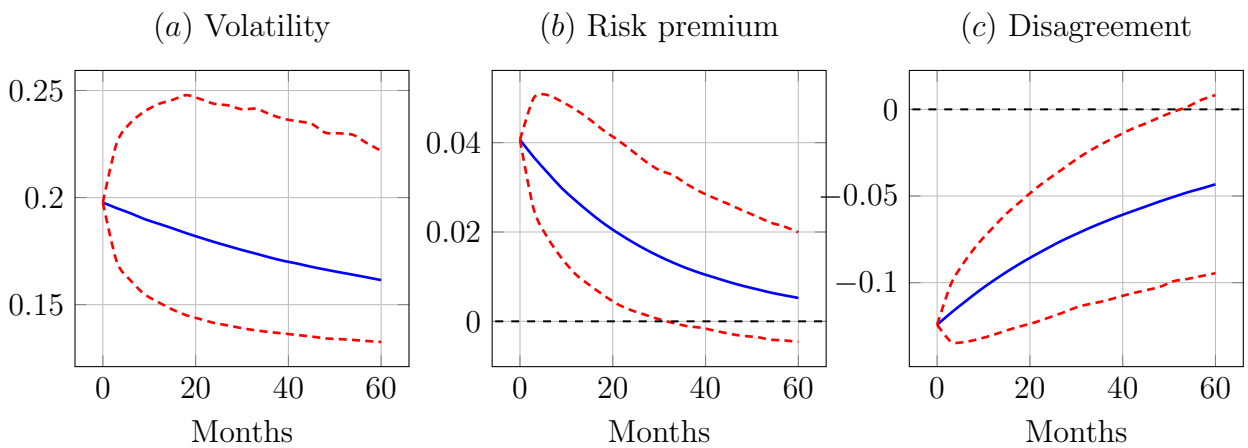

Figure 7: Volatility, risk premium, and disagreement after a recessionary shock
The solid lines show the average values for volatility (panel a), risk premium (panel b), and disagreement $\widehat{\lambda}^B - \widehat{\lambda}^A$ (panel c), averaged across 10,000 simulated 5-year paths at monthly frequency. The dashed lines plot the 5th and 95th percentiles computed across the simulated paths. Parameter values are provided in Table 1. Initial values are $f_0 = -5\%$, $\widehat{\lambda}_0^A = 0.22$, $\widehat{\lambda}_0^B = 0.1$, $\delta_0 = \eta_0 = 1$, and $\gamma_0 = \bar{\gamma}$, where $\bar{\gamma}$ is defined in Appendix A.1.

the equilibrium risk premium. With the benchmark calibration of Table 1, we obtain (from model-simulated data) a 1-month autocorrelation of the risk premium of 0.948, whereas disagreement features a 1-month autocorrelation of 0.992. Given this, disagreement appears to drive the persistence of the risk premium in our model.

Consider the following illustration of the persistent dynamics of the volatility and risk premium after the economy experiences a recessionary shock. Suppose that professional forecasters estimate the growth rate of the economy at -5% (approximately two unconditional standard deviations below the long-term growth $\bar{f} = 2.2\%$). Assume further that the two agents consider two different mean-reversion speeds, $\widehat{\lambda}^A = \bar{\lambda} = 0.22$ and $\widehat{\lambda}^B = 0.1$ (agent B has a less favorable economic outlook and expects a relatively longer recession).

Figure 7 illustrates the evolution of the volatility, risk premium, and disagreement over

five years at monthly frequency. The solid lines represent averages across 10,000 simulations.[27] Panel (a) shows that volatility increases to about 20% (in annualized terms) and then goes down slowly over the next five years (solid line)[28]. The dashed lines are the 5th and 95th percentiles computed across 10,000 simulations. The take-away message of this exercise is that volatility is persistent and features a lot of variability. We have tried several other specifications, including an initial situation without disagreement, and the results are similar. This example illustrates that learning about the mean-reversion speed of the fundamental induces GARCH-like variation in the volatility of stock returns.

Turning to panel (b), the risk premium required by agent B for holding the asset is large and positive (recall that agent B's model has a less favorable economic outlook). One important consideration here is that if we start with an initial situation without disagreement, then the risk premium is close to zero (although it still experiences fluctuations). Disagreement is therefore the main driver of risk premia, whereas uncertainty mostly explains the persistent fluctuations in volatility.

Finally, panel (c) shows the evolution of disagreement (i.e., the difference $\widehat{\lambda}^B - \widehat{\lambda}^A$) over the five-year period of our simulated sample, averaged across 10,000 simulations. Disagreement takes a long time to converge back to zero. Furthermore, the 5th and 95th percentile lines show that disagreement is very volatile. Note also that, although agent B believes initially that the fundamental is more persistent than agent A, there is a chance than in the near future these beliefs will reverse and agent B's model features a less persistent fundamental.

4.4 Survival

The validity of our theoretical results in the long run depends on whether both agents survive in the economy for a sufficiently long period of time. To investigate this, we follow Dumas et al. (2009) and analyze the distribution of the agents' future consumption shares. Consistent with previous sections, we perform our analysis under the probability measure of agent B (i.e., we assume that the beliefs of agent B coincide with the objective—*true*—beliefs).

Figure 8 plots the distribution of the consumption share of agent B resulting from 10,000

[27] The average volatility across simulations and time is 16%, the average risk premium is 0.7%, and the average risk-free rate is 8%. The average risk premium and average risk-free rate would better match their empirical counterparts if agents had habit formation, as in Chan and Kogan (2002), Xiouros and Zapatero (2010), Bhamra and Uppal (2013), and Ehling et al. (2017) among others. Other options are recursive preferences (Epstein and Zin, 1989; Weil, 1989) or dividend leverage (Abel, 1999). As these extensions would not add any insights to the main predictions of our model, we decide to keep the setup simple and focus on the dynamic properties of asset prices.

[28] Given our parameter values, this level of volatility is substantial: the volatility of the dividend growth is 1.2%, the volatility of the fundamental is 2.2% (see Table 1), and the risk aversion is $\alpha = 3$.

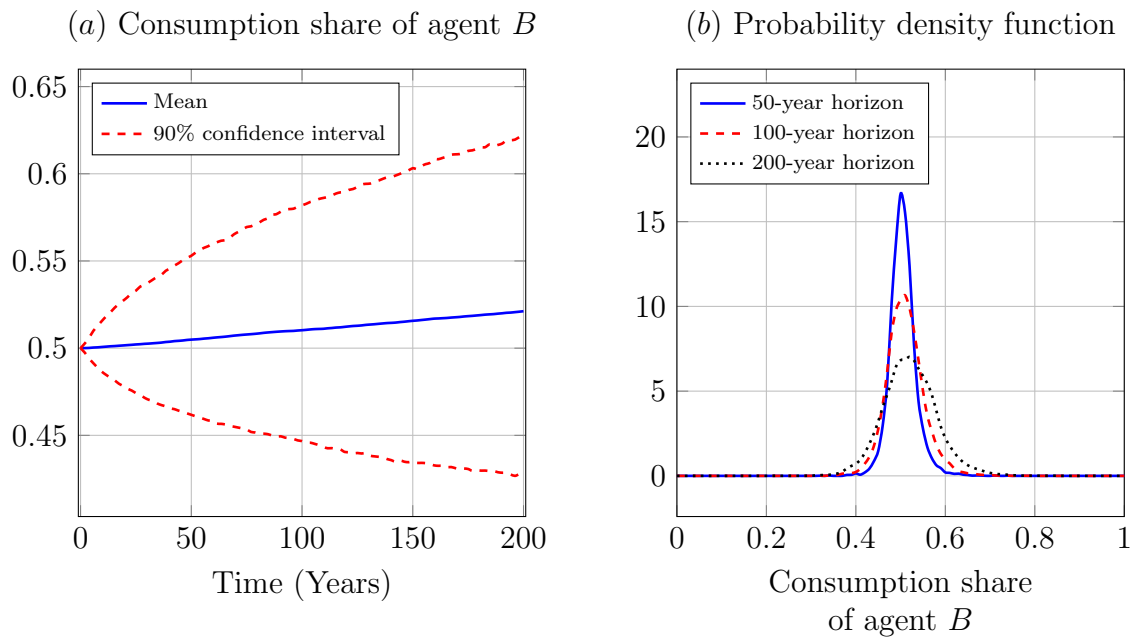

Figure 8: Distribution of the consumption share of agent B

Panel (a) plots the mean (solid line) and the 90% confidence interval (dashed lines) of the consumption share of agent B, resulting from 10,000 simulated economies. Panel (b) plots the probability density function of the consumption share of agent B at a 50-year (solid line), 100-year (dashed line), and 200-year (dotted line) horizon. Parameter values are provided in Table 1. Initial values are $f_0 = \bar{f}$, $\widehat{\lambda}_0^A = \widehat{\lambda}_0^B = \bar{\lambda}$, $\delta_0 = \eta_0 = 1$, and $\gamma_0 = \bar{\gamma}$, where $\bar{\gamma}$ is defined in Appendix A.1.

simulated economies up to 200 years into the future. Panel (a) plots the average consumption share and its 90% confidence interval, whereas panel (b) plots the probability density function for three different horizons. The panels show that both agents' consumption shares remain non-trivial for more than 200 years. Indeed, agent A's misperceptions of the true data-generating process implies that her consumption share is on average 45% after 200 years. The relatively small average loss in agent A's consumption share suggests that agent A's misperceptions are relatively mild.[29] Importantly, both plots show that the distributions of the consumption shares are highly concentrated around 0.5, even at the 200-year horizon. Overall, this analysis suggests that the theoretical results presented in the previous sections are likely to hold for significantly long periods of time, as neither of the agents is likely to vanish in the short term.

[29] Kogan, Ross, Wang, and Westerfield (2016) conduct a thorough analysis of the "market selection hypothesis" and develop necessary and sufficient conditions for survival and price impact. They show that agents with less accurate forecasts maintain a nontrivial consumption share and affect prices if the forecast errors accumulate slowly enough over time. Other references are De Long, Summers, Shleifer, and Waldman (1991), Yan (2008), and Kogan, Ross, Wang, and Westerfield (2006).

5 Conclusion

Understanding what drives stock market volatility remains one of the most critical issues in finance. In this paper, we show that learning about the length of the business cycle generates many of the salient properties of volatility dynamics. When agents need to update their beliefs about the length of the business cycle, uncertainty fluctuates, leading to new asset pricing implications. We characterize these effects in the paper and reach three primary conclusions.

First, this type of learning implies persistent stock-return volatility, which provides a theoretical explanation for GARCH-type processes. Second, the growth rate of the economy governs the "structural uncertainty," which fluctuates and magnifies stock-market volatility in recessions and booms, but also the "structural disagreement," which generates fluctuations in risk premia. Third, the risk-return tradeoff in the economy depends on the value and sign of the structural disagreement. A clear risk-return tradeoff appears when this relationship is properly accounted for.

We test these theoretical predictions with S&P 500 return and U.S. output growth data and find overall support form them. We conclude that these novel implications add to the established literature on learning and heterogeneous beliefs and that they provide plausible explanations for known empirical observations.

A Appendix

A.1 Learning

Following the notations of Liptser and Shiryaev (2001), agent i observes the vector

$$\begin{pmatrix} d\zeta_t \\ df_t \\ ds_t^i \end{pmatrix} = (A_0 + A_1\lambda_t)dt + B_1 dW_t^\lambda + B_2 \begin{pmatrix} dW_t^\delta \\ dW_t^f \\ dW_t^i \end{pmatrix} \tag{34}$$

$$= \left[\begin{pmatrix} f_t - \frac{1}{2}\sigma_\delta^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{f} - f_t \\ 0 \end{pmatrix} \lambda_t \right] dt + \begin{pmatrix} 0 \\ 0 \\ \phi \end{pmatrix} dW_t^\lambda \tag{35}$$

$$+ \begin{pmatrix} \sigma_\delta & 0 & 0 \\ \sigma_f\rho & \sigma_f\sqrt{1-\rho^2} & 0 \\ 0 & 0 & \sqrt{1-\phi^2} \end{pmatrix} \begin{pmatrix} dW_t^\delta \\ dW_t^f \\ dW_t^i \end{pmatrix}, \tag{36}$$

where $i \in \{A, B\}$. The unobservable process λ satisfies

$$d\lambda_t = (a_0 + a_1\lambda_t)dt + b_1 dW_t^\lambda + b_2 \begin{pmatrix} dW_t^\delta \\ dW_t^f \\ dW_t^i \end{pmatrix} \tag{37}$$

$$= \left(\kappa\bar{\lambda} + (-\kappa)\lambda_t\right) dt + \Phi dW_t^\lambda + \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^\delta \\ dW_t^f \\ dW_t^i \end{pmatrix}. \tag{38}$$

Therefore,

$$bob = b_1 b_1' + b_2 b_2' = \Phi^2 \tag{39}$$

$$BoB = B_1 B_1' + B_2 B_2' = \begin{pmatrix} \sigma_\delta^2 & \rho\sigma_f\sigma_\delta & 0 \\ \rho\sigma_f\sigma_\delta & \sigma_f^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{40}$$

$$boB = b_1 B_1' + b_2 B_2' = \begin{pmatrix} 0 & 0 & \phi\Phi \end{pmatrix}. \tag{41}$$

The estimated process defined by $\widehat{\lambda}_t^i = \mathbb{E}^{\mathbb{P}^i}(\lambda_t | \mathscr{O}_t)$ has dynamics

$$d\widehat{\lambda}_t^i = \left(a_0 + a_1\widehat{\lambda}_t^i\right) dt + (boB + \gamma_t A_1')(BoB)^{-1} \left[\begin{pmatrix} d\zeta_t \\ df_t \\ ds_t^i \end{pmatrix} - \left(A_0 + A_1\widehat{\lambda}_t^i\right) dt \right], \tag{42}$$

where the uncertainty γ solves the following Ordinary Differential Equation

$$\frac{d\gamma_t}{dt} = a_1\gamma_t + \gamma_t a_1' + bob - (boB + \gamma_t A_1')(BoB)^{-1}(boB + \gamma_t A_1')'. \tag{43}$$

Consequently,

$$d\widehat{\lambda}_t^i = \kappa\left(\bar{\lambda} - \widehat{\lambda}_t^i\right)dt + \begin{pmatrix} 0 & \frac{(\bar{f}-f_t)\gamma_t}{\sigma_f\sqrt{1-\rho^2}} & \phi\Phi \end{pmatrix}\begin{pmatrix} d\widehat{W}_t^{\delta} \\ d\widehat{W}_t^{fi} \\ d\widehat{W}_t^{si} \end{pmatrix}, \tag{44}$$

where the three Brownians are independent and are defined as follows:

$$d\widehat{W}_t^{\delta} = dW_t^{\delta} \tag{45}$$

$$d\widehat{W}_t^{fi} = \frac{1}{\sigma_f\sqrt{1-\rho^2}}\left[df_t - \widehat{\lambda}_t^i(\bar{f} - f_t)dt - \sigma_f\rho dW_t^{\delta}\right] \tag{46}$$

$$d\widehat{W}_t^{si} = ds_t^i \tag{47}$$

The dynamics of uncertainty are

$$\frac{d\gamma_t}{dt} = \left(1-\phi^2\right)\Phi^2 - 2\kappa\gamma_t - \frac{(\bar{f}-f_t)^2}{\sigma_f^2\left(1-\rho^2\right)}\gamma_t^2. \tag{48}$$

The long-term uncertainty, $\bar{\gamma}$, solves $\left.\frac{d\gamma_t}{dt}\right|_{f_t=\bar{f}} = 0$ and hence satisfies $\bar{\gamma} = \frac{(1-\phi^2)\Phi^2}{2\kappa}$. As the dynamics (48) show, $\bar{\gamma}$ represents an upper bound of uncertainty: when $f_t = \bar{f}$ the last term in (48) vanishes and uncertainty converges to $\bar{\gamma}$. It is only when $f_t \neq \bar{f}$ that uncertainty moves away (downwards) from this upper bound.

A.2 Proof of Proposition 1

Consider the following 4-dimensional Brownian motion defined under the probability measure of agent B:

$$d\widehat{W} = \begin{pmatrix} \widehat{W}^{\delta} \\ \widehat{W}^{fB} \\ \widehat{W}^{sB} \\ \widehat{W}^{sA*} \end{pmatrix} \tag{49}$$

where the first three Brownians are defined in (45), (46) and (47). The last Brownian is defined such that (this ensures that the correlation between $d\widehat{W}^{sA}$ and $d\widehat{W}^{sB}$ is ϕ^2):

$$d\widehat{W}^{sA} = \phi^2 d\widehat{W}^{sB} + \sqrt{1-\phi^4}\,d\widehat{W}^{sA*} \tag{50}$$

Write the dynamics of the vector of state variables under the probability measure of agent B:

$$d\zeta_t = \left(f_t - \frac{1}{2}\sigma_\delta^2\right) dt + \begin{pmatrix}\sigma_\delta & 0 & 0 & 0\end{pmatrix} d\widehat{W}_t \tag{51}$$

$$df_t = \widehat{\lambda}_t^B \left(\bar{f} - f_t\right) dt + \begin{pmatrix}\sigma_f \rho & \sigma_f \sqrt{1-\rho^2} & 0 & 0\end{pmatrix} d\widehat{W}_t \tag{52}$$

$$d\widehat{\lambda}_t^A = \left(\kappa\left(\bar{\lambda} - \widehat{\lambda}_t^A\right) + \frac{\gamma_t\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right)}{\sigma_f^2(1-\rho^2)}\left(\bar{f} - f_t\right)^2\right) dt \tag{53}$$

$$+ \begin{pmatrix}0 & \frac{\gamma_t}{\sigma_f\sqrt{1-\rho^2}}\left(\bar{f} - f_t\right) & \phi^3\Phi & \phi\Phi\sqrt{1-\phi^4}\end{pmatrix} d\widehat{W}_t \tag{54}$$

$$d\widehat{\lambda}_t^B = \kappa\left(\bar{\lambda} - \widehat{\lambda}_t^B\right) dt + \begin{pmatrix}0 & \frac{\gamma_t}{\sigma_f\sqrt{1-\rho^2}}\left(\bar{f} - f_t\right) & \phi\Phi & 0\end{pmatrix} d\widehat{W}_t \tag{55}$$

$$d\gamma_t = \left((1-\phi^2)\Phi - 2\kappa\gamma_t - \frac{(f_t - \bar{f})^2}{\sigma_f^2(1-\rho^2)}\gamma_t^2\right) dt \tag{56}$$

$$d\mu_t = -\frac{1}{2}\frac{\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right)^2\left(\bar{f} - f_t\right)^2}{\sigma_f^2(1-\rho^2)} dt + \begin{pmatrix}0 & -\frac{\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right)\left(\bar{f} - f_t\right)}{\sigma_f\sqrt{1-\rho^2}} & 0 & 0\end{pmatrix} d\widehat{W}_t, \tag{57}$$

The dynamics of disagreement are given by:

$$d\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right) = -\left(\kappa + \frac{\gamma_t\left(\bar{f} - f_t\right)^2}{\sigma_f^2(1-\rho^2)}\right)\left(\widehat{\lambda}_t^B - \widehat{\lambda}_t^A\right) dt + \begin{pmatrix}0 & 0 & \phi\Phi\left(1-\phi^2\right) & \phi\Phi\sqrt{1-\phi^4}\end{pmatrix} d\widehat{W}_t, \tag{58}$$

and thus its conditional variance is

$$2\phi^2\Phi^2\left(1-\phi^2\right). \tag{59}$$

The optimization problem of agent B is

$$\max_{c_{Bt}} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{c_{Bt}^{1-\alpha}}{1-\alpha} dt\right] \tag{60}$$

$$\text{s.t. } \mathbb{E}\left[\int_0^\infty \xi_t^B c_{Bt} dt\right] \leq x_{B0}, \tag{61}$$

where ξ^B denotes the state-price density perceived by agent B and x_{B0} is her initial wealth. Under the probability measure $\mathbb{P}^B$, the problem of agent A is

$$\max_{c_{At}} \mathbb{E}\left[\int_0^\infty \eta_t e^{-\beta t}\frac{c_{At}^{1-\alpha}}{1-\alpha} dt\right] \tag{62}$$

$$\text{s.t. } \mathbb{E}\left[\int_0^\infty \xi_t^B c_{At} dt\right] \leq x_{A0}. \tag{63}$$

Note that the change of measure enters directly the objective function of agent A but not its budget constraint (63). The reason is that the budget constraint depends on the state-price density

perceived by agent B.[30] The first-order conditions are

$$c_{Bt} = \left(\Lambda_B e^{\beta t} \xi_t^B\right)^{-\frac{1}{\alpha}} \tag{64}$$

$$c_{At} = \left(\frac{\Lambda_A}{\eta_t} e^{\beta t} \xi_t^B\right)^{-\frac{1}{\alpha}}, \tag{65}$$

where Λ_A and Λ_B are the Lagrange multipliers associated with the budget constraints of agents A and B, respectively. Summing up agents' optimal consumption policies and imposing market clearing, i.e., $c_{At} + c_{Bt} = \delta_t$, yields the state-price density perceived by agent B

$$\xi_t^B = e^{-\beta t} \delta_t^{-\alpha} \left[\left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha} + \left(\frac{1}{\Lambda_B}\right)^{1/\alpha}\right]^{\alpha}. \tag{66}$$

Substituting the state-price density ξ^B in the optimal consumption policies yields the following consumption sharing rules

$$c_{At} = \omega_{At} \delta_t \tag{67}$$

$$c_{Bt} = \omega_{Bt} \delta_t = (1 - \omega_{At}) \delta_t, \tag{68}$$

where ω_{it} denotes agent i's share of consumption at time t for $i \in \{A, B\}$. Agent A's share of consumption satisfies

$$\omega_{At} = \frac{\left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha}}{\left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha} + \left(\frac{1}{\Lambda_B}\right)^{1/\alpha}}. \tag{69}$$

The evolution of agent B's consumption share under her own probability measure follows

$$\frac{d\omega_{Bt}}{\omega_{Bt}} = \frac{1}{2} \frac{(1 - \omega_{Bt})(1 + \alpha - 2\omega_{Bt})}{\alpha^2 \sigma_f^2 (1 - \rho^2)} \mathcal{D}_t^2 dt + \frac{1 - \omega_{Bt}}{\alpha \sigma_f \sqrt{1 - \rho^2}} \mathcal{D}_t d\widehat{W}_t^{fB}. \tag{70}$$

Notice that the drift is always positive (as long as there is disagreement). This is also the case for the consumption share of agent A, under her own probability measure.

As in Yan (2008) and Dumas et al. (2009), we assume that the coefficient of relative risk aversion

[30] Alternatively, we could have defined the state-price density perceived by agent A, ξ^A. Then, we would have $\mathbb{E}^A\left[\xi^A \mathbf{1}_x\right] = \mathbb{E}^B\left[\eta \xi^A \mathbf{1}_x\right] = \mathbb{E}^B\left[\xi^B \mathbf{1}_x\right]$ for any event x. That is, $\xi^B = \eta \xi^A$.

α is an integer. In this case, the state-price density at time u satisfies

$$\xi_u^B = e^{-\beta u}\delta_u^{-\alpha}\left[\left(\frac{1}{\Lambda_B}\right)^{1/\alpha} + \left(\frac{\eta_u}{\Lambda_A}\right)^{1/\alpha}\right]^{\alpha} \tag{71}$$

$$= e^{-\beta u}\delta_u^{-\alpha}\frac{1}{\Lambda_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{\eta_u\Lambda_B}{\Lambda_A}\right)^{\frac{j}{\alpha}} \tag{72}$$

$$= e^{-\beta u}\delta_u^{-\alpha}\frac{1}{\Lambda_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{1}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\eta_t\Lambda_B}{\Lambda_A}\right)^{\frac{j}{\alpha}}\eta_u^{\frac{j}{\alpha}} \tag{73}$$

$$= e^{-\beta u}\delta_u^{-\alpha}\frac{1}{\Lambda_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega_{At}}{1-\omega_{At}}\right)^{j}, \tag{74}$$

where the last equality comes from the fact that

$$\omega_{At} = \frac{\left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha}}{\left(\frac{1}{\Lambda_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha}} \tag{75}$$

$$1-\omega_{At} = \frac{\left(\frac{1}{\Lambda_B}\right)^{1/\alpha}}{\left(\frac{1}{\Lambda_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha}}, \tag{76}$$

and consequently

$$\left(\frac{\eta_t\Lambda_B}{\Lambda_A}\right)^{\frac{1}{\alpha}} = \frac{\omega_{At}}{1-\omega_{At}}. \tag{77}$$

Rewriting Equation (76) yields

$$\left(\frac{1}{\Lambda_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\Lambda_A}\right)^{1/\alpha} = \left(\frac{1}{1-\omega_{At}}\right)\left(\frac{1}{\Lambda_B}\right)^{\frac{1}{\alpha}}, \tag{78}$$

and thus

$$\xi_t^B = e^{-\beta t}\delta_t^{-\alpha}\left(\frac{1}{1-\omega_{At}}\right)^{\alpha}\frac{1}{\Lambda_B} = \frac{1}{\Lambda_B}e^{-\beta t}(\delta_t\omega_{Bt})^{-\alpha}, \tag{79}$$

which is Equation (20) in the text. Thus, the dynamics for the stochastic discount follow:

$$\frac{d\xi_t}{\xi_t} = -\Bigg[\beta + \alpha f_t - \frac{1}{2}\alpha(\alpha+1)\sigma_\delta^2 \tag{80}$$

$$+\alpha\frac{1}{2}\frac{(1-\omega_{Bt})(1+\alpha-2\omega_{Bt})}{\alpha^2\sigma_f^2(1-\rho^2)}\mathcal{D}_t^2 - \frac{1}{2}\alpha(\alpha+1)\frac{(1-\omega_{Bt})^2}{\alpha^2\sigma_f^2(1-\rho^2)}\mathcal{D}_t^2\Bigg]\,dt \tag{81}$$

$$-\underbrace{\begin{pmatrix}\alpha\sigma_\delta & \frac{\mathcal{D}_t}{\sigma_f\sqrt{1-\rho^2}}\omega_{At} & 0 & 0\end{pmatrix}}_{\text{Market price of risk}}\begin{pmatrix}d\widehat{W}_t^{\delta}\\ d\widehat{W}_t^{fB}\\ d\widehat{W}_t^{sB}\\ d\widehat{W}_t^{sA*}\end{pmatrix}, \tag{82}$$

and the risk-free rate is given by

$$r_t = \beta + \alpha f_t - \frac{1}{2}\alpha(\alpha+1)\sigma_\delta^2 + \frac{1}{2}\frac{\alpha-1}{\alpha}\frac{\mathcal{D}_t^2}{\sigma_f^2(1-\rho^2)}\omega_{At}\omega_{Bt}. \tag{83}$$

The price-dividend ratio satisfies

$$\frac{S_t}{\delta_t} = \mathbb{E}_t\left(\int_t^\infty \frac{\xi_u^B}{\xi_t^B}\frac{\delta_u}{\delta_t}du\right) \tag{84}$$

$$\overset{(74)\ (78)\ (79)}{=} \mathbb{E}_t\left(\int_t^\infty \frac{e^{-\beta u}\delta_u^{-\alpha}\frac{1}{\Lambda_B}\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega_{At}}{1-\omega_{At}}\right)^j}{e^{-\beta t}\left(\frac{1}{1-\omega_{At}}\right)^\alpha\frac{1}{\Lambda_B}\delta_t^{-\alpha}}\frac{\delta_u}{\delta_t}du\right) \tag{85}$$

$$= \mathbb{E}_t\left(\int_t^\infty e^{-\beta(u-t)}\frac{\sum_{j=0}^{\alpha}\binom{\alpha}{j}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\omega_{At}}{1-\omega_{At}}\right)^j}{\left(\frac{1}{1-\omega_{At}}\right)^\alpha}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha}du\right) \tag{86}$$

$$= \sum_{j=0}^{\alpha}\binom{\alpha}{j}\omega_{At}^j(1-\omega_{At})^{\alpha-j}\,\mathbb{E}_t\left(\int_t^\infty e^{-\beta(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\frac{j}{\alpha}}\left(\frac{\delta_u}{\delta_t}\right)^{1-\alpha}du\right). \tag{87}$$

Let us define the function $F_j(Z)$ as follows

$$F_j(Z_t) \equiv \mathbb{E}_t\left[\int_t^\infty e^{-\beta(u-t)}\left(\frac{\eta_u}{\eta_t}\right)^{\epsilon}\left(\frac{\delta_u}{\delta_t}\right)^{\chi}du\right], \tag{88}$$

where $\epsilon = \frac{j}{\alpha},\ j = 0,\ldots,\alpha$, $\chi = 1-\alpha$, and $Z \equiv \left(f, \widehat{\lambda}^A, \widehat{\lambda}^B, \gamma\right)^\top$ is a vector of state-variables that does not comprise $\zeta = \log\delta$ and $\mu = \log\eta$.

Using these notations, the price-dividend ratio satisfies

$$\frac{S_t}{\delta_t} = \sum_{j=0}^{\alpha}\binom{\alpha}{j}\omega_{At}^j(1-\omega_{At})^{\alpha-j}F_j(Z_t) \tag{89}$$

$$= \sum_{j=0}^{\alpha}\binom{\alpha}{j}\omega_{At}^j\omega_{Bt}^{\alpha-j}F_j(Z_t), \tag{90}$$

which is Equation (23) in Proposition 1.

A.3 Solution method (Chebyshev Collocation)

We have

$$\left(\frac{\eta_u}{\eta_t}\right)^{\epsilon} = e^{-\frac{1}{2}\int_t^u \epsilon\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right)^2 ds-\int_t^u\begin{pmatrix}0 & \epsilon\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right) & 0 & 0\end{pmatrix}d\widehat{W}_t} \tag{91}$$

$$\left(\frac{\delta_u}{\delta_t}\right)^{\chi} = e^{\int_t^u \chi\left(f_s-\frac{1}{2}\sigma_\delta^2\right)ds+\int_t^u\begin{pmatrix}\chi\sigma_\delta & 0 & 0 & 0\end{pmatrix}d\widehat{W}_t}, \tag{92}$$

where ϵ and χ are some constants. Therefore,

$$\left(\frac{\eta_u}{\eta_t}\right)^{\epsilon}\left(\frac{\delta_u}{\delta_t}\right)^{\chi} = e^{\int_t^u\left[\chi\left(f_s-\frac{1}{2}\sigma_\delta^2\right)-\frac{1}{2}\epsilon\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right)^2+\frac{1}{2}\left(\chi^2\sigma_\delta^2+\epsilon^2\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right)^2\right)\right]ds} \tag{93}$$

$$\times\, e^{-\frac{1}{2}\int_t^u\left(\chi^2\sigma_\delta^2+\epsilon^2\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right)^2\right)ds-\int_t^u\begin{pmatrix}-\chi\sigma_\delta & \epsilon\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right) & 0 & 0\end{pmatrix}d\widehat{W}_t}. \tag{94}$$

Note that last term of the first row cancels the first term of the second row. Importantly, the second row defines a change of measure. The change of measure is

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}^B}\right|_{\mathscr{O}_t} \equiv \nu_t = e^{-\frac{1}{2}\int_t^u\left(\chi^2\sigma_\delta^2+\epsilon^2\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right)^2\right)ds-\int_t^u\begin{pmatrix}-\chi\sigma_\delta & \epsilon\left(\frac{\left(\widehat{\lambda}_s^B-\widehat{\lambda}_s^A\right)\left(\bar{f}-f_s\right)}{\sigma_f\sqrt{1-\rho^2}}\right) & 0 & 0\end{pmatrix}d\widehat{W}_t}, \tag{95}$$

where the $\bar{\mathbb{P}}$-Brownian motion $\bar{W}$ is defined as

$$d\bar{W}_t = d\widehat{W}_t + y_t dt \tag{96}$$

$$y_t = \begin{pmatrix}-\chi\sigma_\delta & \epsilon\left(\frac{\left(\widehat{\lambda}_t^B-\widehat{\lambda}_t^A\right)\left(\bar{f}-f_t\right)}{\sigma_f\sqrt{1-\rho^2}}\right) & 0 & 0\end{pmatrix}^\top. \tag{97}$$

Rewriting the problem under the probability measure $\bar{\mathbb{P}}$ yields

$$F(Z_t) \equiv \bar{\mathbb{E}}_t\left[\int_t^\infty e^{\int_t^u X_s ds}du\right], \tag{98}$$

where $X_t = -\beta + \frac{1}{2}\frac{\left(\widehat{\lambda}_t^B-\widehat{\lambda}_t^A\right)^2\left(\bar{f}-f_t\right)^2}{\sigma_f^2(1-\rho^2)}\left(\epsilon^2-\epsilon\right)+\chi\left(f_t-\frac{1}{2}\sigma_\delta^2\right)+\frac{1}{2}\chi^2\sigma_\delta^2$. For notational ease, we drop the index j when defining the function $F(.)$ by keeping in mind that $\epsilon = \frac{j}{\alpha}$ and $\chi = 1-\alpha$ in our setup. We now transform this expression to obtain a $\bar{\mathbb{P}}$-martingale. We have

$$F(Z_t)e^{\int_0^t X_S ds} + \int_0^t e^{\int_0^u X_s ds}du = \bar{\mathbb{E}}_t\left[\int_0^\infty e^{\int_0^u X_s ds}du\right] \equiv \bar{M}_t, \tag{99}$$

where $\bar{M}$ is a $\bar{\mathbb{P}}$-martingale. Applying Itô's lemma to the martingale $\bar{M}$ and setting its drift to zero yields the following Partial Differential Equation for the function $F(Z)$

$$\mathscr{L}^Z F(Z) + F(Z)X(Z) + 1 = 0, \tag{100}$$

where $\mathscr{L}^Z$ is the $\bar{\mathbb{P}}$-infinitesimal generator with respect to the vector of state variables $Z \equiv (f, \widehat{\lambda}^A, \widehat{\lambda}^B, \gamma)^\top$.

PDE (100) is solved numerically using the Chebyshev collocation method (Judd, 1998). This method consists in approximating the function $F(Z)$ as follows:

$$F(Z) \approx P(Z) = \sum_{i=0}^{I}\sum_{j=0}^{J}\sum_{k=0}^{K}\sum_{l=0}^{L} a_{i,j,k,l} T_i(f)\, T_j\left(\widehat{\lambda}^A\right) T_k\left(\widehat{\lambda}^B\right) T_l(\gamma),$$

where T_m is the Chebyshev polynomial of order m. We mesh the roots of the Chebyshev polynomials of order $I+1$, $J+1$, $K+1$, and $L+1$ to obtain the interpolation nodes. Furthermore, these interpolation nodes are such that they cover the 99% confidence interval of each state variable. Substituting $P(Z)$ and its partial derivatives in the PDE and evaluating the latter at the interpolation nodes yields a system of $(I+1)\times(J+1)\times(K+1)\times(L+1)$ equations with $(I+1)\times(J+1)\times(K+1)\times(L+1)$ unknowns (the coefficients $a_{i,j,k,l}$). This system is solved numerically by minimizing the sum of squared PDE residuals at the interpolation nodes.

This method is very accurate even when the order of the Chebyshev polynomials is relatively low. Indeed, choosing $I = J = K = L = 4$ yields $5^4 = 625$ interpolation nodes. In our setup, the sum (over these 625 interpolation nodes) of squared PDE residuals is of order 10^{-33} only. Furthermore, simulations show that the orders of the 1-percentile, median, and 99-percentile of squared residuals are 10^{-19}, 10^{-15}, and 10^{-12}, respectively. This provides evidence that the method is accurate.

A.4 Calibration to the U.S. economy (Maximum Likelihood)

The log dividend growth rate and the fundamental are discretized as follows:

$$\log(\delta_{t+\Delta}/\delta_t) \approx \left(f_t - \frac{1}{2}\sigma_\delta^2\right)\Delta + \sigma_\delta\sqrt{\Delta}\epsilon_{1,t+\Delta} \tag{101}$$

$$f_{t+\Delta} \approx e^{-\widehat{\lambda}_t^B\Delta} f_t + \bar{f}\left(1 - e^{-\widehat{\lambda}_t^B\Delta}\right) + \frac{\sigma_f\rho}{\sqrt{2\widehat{\lambda}_t^B}}\left(1 - e^{-2\widehat{\lambda}_t^B\Delta}\right)\epsilon_{1,t+\Delta} \tag{102}$$

$$+ \frac{\sigma_f\sqrt{1-\rho^2}}{\sqrt{2\widehat{\lambda}_t^B}}\left(1 - e^{-2\widehat{\lambda}_t^B\Delta}\right)\epsilon_{2,t+\Delta}, \tag{103}$$

where $\epsilon_1, \epsilon_2 \sim N(0,1)$ are i.i.d and $\Delta = 1/4$ because our dataset is at quarterly frequency. We fit $\log(\delta_{t+\Delta}/\delta_t)$ and f_t to the realized real GDP growth over the current quarter and median real GDP growth forecast for the current quarter, respectively.

In the same spirit, Agent A's forecast at horizon $s-t$, $f^A_{t,s-t}$, and Agent B's forecast at horizon $s-t$, $f^B_{t,s-t}$, are discretized as follows:

$$f^A_{t,s-t} = E^A_t(f_s) \approx e^{-\widehat{\lambda}_t^A(s-t)} f_t + \bar{f}\left(1 - e^{-\widehat{\lambda}_t^A(s-t)}\right) \tag{104}$$

$$f^B_{t,s-t} = E^B_t(f_s) \approx e^{-\widehat{\lambda}_t^B(s-t)} f_t + \bar{f}\left(1 - e^{-\widehat{\lambda}_t^B(s-t)}\right). \tag{105}$$

Applying Itô's lemma to Equations (104) and (105), substituting f_t by its value extracted from Equations (104) and (105), and setting $s - t = \Delta = 1/4$ yields the following dynamics for the 1-quarter-ahead forecasts computed by Agent A, $f^A_{t,\Delta}$, and Agent B, $f^B_{t,\Delta}$

$$df^A_{t,\Delta} = a_1(f^A_{t,\Delta}, \widehat{\lambda}^A_t, \widehat{\lambda}^B_t, \gamma_t)dt + b_1(f^A_{t,\Delta}, \widehat{\lambda}^A_t, \gamma_t)d\widehat{W}_t, \tag{106}$$

$$df^B_{t,\Delta} = a_2(f^B_{t,\Delta}, \widehat{\lambda}^B_t, \gamma_t)dt + b_2(f^B_{t,\Delta}, \widehat{\lambda}^B_t, \gamma_t)d\widehat{W}_t, \tag{107}$$

where the drifts $a_1(.), a_2(.) \in \mathbb{R}$ and the diffusions $b_1(.), b_2(.) \in \mathbb{R}^4$ are available upon request.

Discretizing the dynamics in (106) and (107) yields

$$f^A_{t+\Delta,\Delta} = f^A_{t,\Delta} + a_1(f^A_{t,\Delta}, \widehat{\lambda}^A_t, \widehat{\lambda}^B_t, \gamma_t)\Delta + b_1(f^A_{t,\Delta}, \widehat{\lambda}^A_t, \gamma_t)\sqrt{\Delta}\begin{pmatrix}\epsilon_{1,t+\Delta}\\ \epsilon_{2,t+\Delta}\\ \epsilon_{3,t+\Delta}\\ \epsilon_{4,t+\Delta}\end{pmatrix}, \tag{108}$$

$$f^B_{t+\Delta,\Delta} = f^B_{t,\Delta} + a_2(f^B_{t,\Delta}, \widehat{\lambda}^B_t, \gamma_t)\Delta + b_2(f^B_{t,\Delta}, \widehat{\lambda}^B_t, \gamma_t)\sqrt{\Delta}\begin{pmatrix}\epsilon_{1,t+\Delta}\\ \epsilon_{2,t+\Delta}\\ \epsilon_{3,t+\Delta}\\ \epsilon_{4,t+\Delta},\end{pmatrix} \tag{109}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \sim N(0,1)$ are i.i.d.

We fit $f^A_{t,\Delta}$ and $f^B_{t,\Delta}$ to the 75th percentile of the 1-quarter-ahead real GDP growth forecast and to the 25th percentile of the 1-quarter-ahead real GDP growth forecast, respectively.

The four time series, namely realized GDP growth, median of the current quarter GDP growth forecast, 75th percentile of the 1-quarter-ahead GDP growth forecast, and 25th percentile of the 1-quarter-ahead GDP growth forecast, are available at quarterly frequency from Q4:1968 to Q2:2016 and are obtained from the Federal Reserve Bank of Philadelphia's website.

Maximizing the log likelihood function implied by (101), (103), (108), and (109) yields the parameters and standard errors reported in Table 1. The estimation also provides the historical model-implied time series of the mean-reversion speed of agent A, $\widehat{\lambda}^{A,e}$, the mean-reversion speed of agent B, $\widehat{\lambda}^{B,e}$, and the uncertainty, γ^e. The historical time series of the fundamental consists of the median of the current quarter GDP growth forecast and is denoted by $f^e \equiv f$.

References

Abel, A. B. (1999). Risk premia and term premia in general equilibrium. *Journal of Monetary Economics 43*(1), 3–33.

Alti, A. and P. C. Tetlock (2014). Biased beliefs, asset prices, and investment: A structural approach. *The Journal of Finance 69*(1), 325–361.

Andrade, P., R. K. Crump, S. Eusepi, and E. Moench (2017). Noisy information and fundamental disagreement. *Forthcoming, Journal of Monetary Economics*.

Bachmann, R., S. Elstner, and E. R. Sims (2013, April). Uncertainty and Economic Activity: Evidence from Business Survey Data. *American Economic Journal: Macroeconomics 5*(2), 217–49.

Baker, S. D., B. Hollifield, and E. Osambela (2015). Disagreement, speculation, and aggregate investment. *Journal of Financial Economics*, –.

Bansal, R., D. Kiku, and A. Yaron (2010, September). Long run risks, the macroeconomy, and asset prices. *American Economic Review 100*(2), 542–46.

Bansal, R., D. Kiku, and A. Yaron (2012). An empirical evaluation of the long-run risks model for asset prices. *Critical Finance Review 1, no. 1*, 183–221.

Bansal, R. and A. Yaron (2004, 08). Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance 59*(4), 1481–1509.

Barberis, N., M. Huang, and T. Santos (2001, February). Prospect theory and asset prices. *The Quarterly Journal of Economics 116*(1), 1–53.

Barinov, A. (2014, 3). Analyst disagreement and aggregate volatility risk. *Journal of Financial and Quantitative Analysis FirstView*, 1–43.

Basak, S. (2000, January). A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk. *Journal of Economic Dynamics and Control 24*(1), 63–95.

Basak, S. (2005, November). Asset pricing with heterogeneous beliefs. *Journal of Banking & Finance 29*(11), 2849–2881.

Beeler, J. and J. Y. Campbell (2012). The long-run risks model and aggregate asset prices: An empirical assessment. *Critical Finance Review 1*, 141–182.

Benth, F. E. and A. Khedher (2016). *Weak Stationarity of Ornstein-Uhlenbeck Processes with Stochastic Speed of Mean Reversion*, pp. 153–189. Cham: Springer International Publishing.

Bernanke, B. S. (2013, April). *The Federal Reserve and the Financial Crisis*, Volume 1 of *Economics Books*. Princeton University Press.

Berrada, T. (2006). Incomplete information, heterogeneity, and asset pricing. *Journal of Financial Econometrics 4*(1), 136–160.

Berrada, T. (2009). Bounded rationality and asset pricing with intermediate consumption*. *Review of Finance 13*(4), 693–725.

Bhamra, H. S. and R. Uppal (2013). Asset prices with heterogeneity in preferences and beliefs. *Review of Financial Studies*.

Bloom, N. (2009, 05). The impact of uncertainty shocks. *Econometrica 77*(3), 623–685.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S. J. Terry (2014, March). Really Uncertain Business Cycles. Working Papers 14-18, Center for Economic Studies, U.S. Census Bureau.

Bollerslev, T. (1986, April). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 31*(3), 307–327.

Brandt, M. W. and Q. Kang (2004). On the relationship between the conditional mean and volatility of stock returns: A latent var approach. *Journal of Financial Economics 72*(2), 217–257.

Brennan, M. J. (1998). The role of learning in dynamic portfolio decisions. *European Finance Review 1*(3), 295–306.

Brennan, M. J. and Y. Xia (2001). Stock price volatility and equity premium. *Journal of Monetary Economics 47*(2), 249 – 283.

Buraschi, A. and A. Jiltsov (2006). Model uncertainty and option markets with heterogeneous beliefs. *The Journal of Finance 61*(6), pp. 2841–2897.

Buraschi, A., F. Trojani, and A. Vedolin (2014). When uncertainty blows in the orchard: Comovement and equilibrium volatility risk premia. *The Journal of Finance 69*(1), 101–137.

Buraschi, A. and P. Whelan (2013). Term structure models and differences in beliefs. *Working Paper*.

Cagetti, M., L. P. Hansen, T. Sargent, and N. Williams (2002, March). Robustness and Pricing with Uncertain Growth. *Review of Financial Studies 15*(2), 363–404.

Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: A consumption-based explanation of aggregate stock market beha. *Journal of Political Economy 107*, 205–251.

Carlin, B. I., F. A. Longstaff, and K. Matoba (2014). Disagreement and asset prices. *Journal of Financial Economics 114*(2), 226 – 238.

Chan, Y. L. and L. Kogan (2002, December). Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices. *Journal of Political Economy 110*(6), 1255–1285.

Cochrane, J. H. (2008). The dog that did not bark: A defense of return predictability. *Review of Financial Studies 21*(4), 1533–1575.

Collin-Dufresne, P., M. Johannes, and L. A. Lochstoer (2015, December). Parameter learning in general equilibrium: The asset pricing implications. *American Economic Review* (19705).

Cox, J. C. and C.-f. Huang (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory 49*(1), 33 – 83.

Croce, M. M., M. Lettau, and S. C. Ludvigson (2014). Investor information, long-run risk, and the term structure of equity*. *Review of Financial Studies*.

Cvitanic, J., E. Jouini, S. Malamud, and C. Napp (2011). Financial markets equilibrium with heterogeneous agents*. *Review of Finance 16*(1), 285.

Cvitanic, J. and S. Malamud (2011). Price impact and portfolio impact. *Journal of Financial Economics 100*(1), 201 – 225.

David, A. (1997, December). Fluctuating confidence in stock markets: Implications for returns and volatility. *Journal of Financial and Quantitative Analysis 32*(04), 427–462.

David, A. (2008). Heterogeneous beliefs, speculation and the equity premium. *Journal of Finance 63*, 41–83.

David, A. and P. Veronesi (2002). Option Prices with Uncertain Fundamental: Theory and Evidence on the Dynamics of Implief Volatilities and Over-reaction in the Options Markets. *Working Paper*.

De Long, J. B., L. H. Summers, A. Shleifer, and R. Waldman (1991, January). The Survival of Noise Traders in Financial Markets. *The Journal of Business 64*(1), 1–19.

Detemple, J. and S. Murthy (1994, April). Intertemporal asset pricing with heterogeneous beliefs. *Journal of Economic Theory 62*(2), 294–320.

Detemple, J. B. (1986, June). Asset pricing in a production economy with incomplete information. *Journal of Finance 41*(2), 383–91.

Detemple, J. B. (1991, July). Further results on asset pricing with incomplete information. *Journal of Economic Dynamics and Control 15*(3), 425–453.

Dothan, M. U. and D. Feldman (1986, June). Equilibrium interest rates and multiperiod bonds in a partially observable economy. *Journal of Finance 41*(2), 369–82.

Dumas, B., A. Kurshev, and R. Uppal (2009, 04). Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility. *Journal of Finance 64*(2), 579–629.

Dumas, B., K. K. Lewis, and E. Osambela (2016). Differences of opinion and international equity markets. *Review of Financial Studies*.

Ehling, P., M. Gallmeyer, C. Heyerdahl-Larsen, and P. Illeditsch (2017). Disagreement about inflation and the yield curve. *Forthcoming, Journal of Financial Economics*.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica 50*(4), pp. 987–1007.

Epstein, L. G. and S. E. Zin (1989, July). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*(4), 937–69.

Gennotte, G. (1986, July). Optimal portfolio choice under incomplete information. *Journal of Finance 41*(3), 733–46.

Glosten, L. R., R. Jagannathan, and D. E. Runkle (1993, December). On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance 48*(5), 1779–1801.

Hamilton, J. D., E. S. Harris, J. Hatzius, and K. D. West (2015). The Equilibrium Real Funds Rate: Past, Present and Future. *Working Paper*.

Harris, M. and A. Raviv (1993). Differences of opinion make a horse race. *Review of Financial Studies 6*(3), 473–506.

Hirshleifer, D., J. Li, and J. Yu (2015). Asset pricing in production economies with extrapolative expectations. *Journal of Monetary Economics 76*, 87–106.

Howard, G., R. Martin, and B. A. Wilson (2011). Are recoveries from banking and financial crises really so different? International Finance Discussion Papers 1037, Board of Governors of the Federal Reserve System (U.S.).

Ju, N. and J. Miao (2012). Ambiguity, Learning, and Asset Returns. *Econometrica 80*, 559–591.

Judd, K. L. (1998). *Numerical Methods in Economics*. MIT Press.

Jurado, K., S. C. Ludvigson, and S. Ng (2015, March). Measuring Uncertainty. *American Economic Review 105*(3), 1177–1216.

Kandel, E. and N. D. Pearson (1995, August). Differential interpretation of public signals and trade in speculative markets. *Journal of Political Economy 103*(4), 831–72.

Kogan, L., S. A. Ross, J. Wang, and M. M. Westerfield (2006). The price impact and survival of irrational traders. *The Journal of Finance 61*(1), pp. 195–229.

Kogan, L., S. A. Ross, J. Wang, and M. M. Westerfield (2016). Market selection. *Journal of Economic Theory*.

Li, T. (2007, May). Heterogeneous beliefs, asset prices, and volatility in a pure exchange economy. *Journal of Economic Dynamics and Control 31*(5), 1697–1727.

Liptser, R. S. and A. N. Shiryaev (2001). *Statistics of Random Processes II*. Springer Verlag, New York.

Liu, J., J. Pan, and T. Wang (2005). An equilibrium model of rare-event premia and its implication for option smirks. *Review of Financial Studies 18*(1), 131–164.

McQueen, G. and K. Vorkink (2004). Whence garch? a preference-based explanation for conditional volatility. *Review of Financial Studies 17*(4), 915–949.

Newey, W. K. and K. D. West (1987, May). A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica 55*(3), 703–08.

Osambela, E. (2015). Differences of Opinion, Endogenous Liquidity, and Asset Prices. *Review of Financial Studies 28*(7), 1914–1959.

Pastor, L. and P. Veronesi (2006, July). Was there a nasdaq bubble in the late 1990s? *Journal of Financial Economics 81*(1), 61–100.

Patton, A. J. and A. Timmermann (2010, October). Why do forecasters disagree? lessons from the term structure of cross-sectional dispersion. *Journal of Monetary Economics 57*(7), 803–820.

Reinhart, C. M. and K. S. Rogoff (2009). The aftermath of financial crises. *American Economic Review 99*(2), 466–72.

Rossi, A. G. and A. Timmermann (2015). Modeling covariance risk in merton's icapm. *Review of Financial Studies*, hhv015.

Scheinkman, J. A. and W. Xiong (2003). Overconfidence and speculative bubbles. *The Journal of Political Economy 111*(6), 1183–1219.

Schwert, G. W. (1989, December). Why does stock market volatility change over time? *Journal of Finance 44*(5), 1115–53.

Shiller, R. J. (1981, June). Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review 71*(3), 421–36.

Summers, L. H. (2014, April). U.S. Economic Prospects: Secular Stagnation, Hysteresis, and the Zero Lower Bound. *Business Economics 49*(2), 65–73.

Timmermann, A. (2001, July). Structural breaks, incomplete information, and stock prices. *Journal of Business & Economic Statistics 19*(3), 299–314.

Van Binsbergen, J. H. and R. S. J. Koijen (2010). Predictive regressions: A present-value approach. *The Journal of Finance 65*(4), 1439–1471.

Veronesi, P. (1999). Stock market overreaction to bad news in good times: A rational expectations equilibrium model. *Review of Financial Studies 12*(5), 975–1007.

Veronesi, P. (2000, April). How does information quality affect stock returns? *Journal of Finance 55*(2), 807–837.

Weil, P. (1989, November). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics 24*(3), 401–421.

Whelan, P. (2014). Model disagreement and real bonds. *Working Paper*.

Xia, Y. (2001, 02). Learning about Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation. *Journal of Finance 56*(1), 205–246.

Xiong, W. and H. Yan (2010, April). Heterogeneous expectations and bond markets. *Review of Financial Studies 23*(4), 1433–1466.

Xiouros, C. and F. Zapatero (2010, August). The Representative Agent of an Economy with External Habit Formation and Heterogeneous Risk Aversion. *Review of Financial Studies 23*(8), 3017–3047.

Yan, H. (2008, November). Natural Selection in Financial Markets: Does It Work? *Management Science 54*(11), 1935–1950.

Zapatero, F. (1998, April). Effects of financial innovations on market volatility when beliefs are heterogeneous. *Journal of Economic Dynamics and Control 22*(4), 597–626.