Abstract

We study a pure exchange economy with incomplete information in which two agents are uncertain about the length of the business cycle. That is, the agents do not question whether the economy is growing or not, but instead continuously estimate how long economic cycles will last. Learning about persistence, as opposed to learning about the growth rate itself, helps to rationalize several salient properties of stock market volatility. Uncertainty endogenously fluctuates and generates high and persistent stock-return volatility in recessions and booms. Disagreement among agents earns a risk premium and becomes the channel through which uncertainty is priced in the economy.

Keywords: Learning, Uncertainty, Disagreement, Volatility, Risk Premium

JEL Classification: D51, D83, G12, G14

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1 Introduction

Asset return volatility is known to be excessively high relative to the volatility of fundamentals (Shiller, 1981), to be time-varying and predictable (Engle, 1982), to increase during recessions (Schwert, 1989) but also during periods of rapid technological progress (Pastor and Veronesi, 2006). So far, no consensus has been reached as to which model can simultaneously explain all of these empirical regularities. The purpose of this paper is to offer such a model.

We study a pure exchange economy with incomplete information in which two agents are uncertain about the length of the business cycle. That is, agents do not question whether the economy is growing or not—the usual learning exercise studied in the literature—but instead continuously wonder how long economic cycles will last. We show that learning about the persistence of economic growth differs from learning about economic growth itself and helps account for a wide range of empirical observations regarding stock-return volatility.

The first implication is that learning about persistence generates time-varying uncertainty. This introduces a main object of interest in our model, which we call structural uncertainty. It is defined as the product of two quantities: the uncertainty about growth persistence and the difference between the growth rate and its long-term mean. Intuitively, structural uncertainty is a function of how much uncertainty is perceived by agents and the degree to which the economy is in an expansion or contraction. This implies that the way in which structural uncertainty impacts the price of risk, risk premia, and the volatility of stock returns depends on the growth rate of the economy.

We show that structural uncertainty generates time-varying and persistent stock market volatility. When the economy is going through recessions and booms, agents face significant structural uncertainty, which further increases stock return volatility. For instance, during the Great Recession of 2007-08 there was significant uncertainty about how long it will last, with economists and policymakers debating the odds of a deep versus a mild recession. This particular type of uncertainty, present in our model, generates high volatility when the growth rate is far-away from normal times. Thus, our model offers an alternative view, different from other models of learning in which uncertainty is high when agents perceive the economy to be “in-between” a discrete set of growth states (e.g., Veronesi, 1999, 2000).\(^1\)

Because agents use different sources of information to estimate the length of business cycles, this introduces an additional key variable, which we call structural disagreement. It is defined as the product of two quantities: the degree to which agents’ estimates of the mean-reversion speed of the fundamental diverge from each other and the difference between the fundamental and its long-term mean. Given this, structural disagreement is enhanced during booms and recessions. It impacts the market price of risk by increasing each agent’s consumption share risk, and is associated with a positive risk premium that grows as the economy goes through booms or recessions. When disagreement is high, the risk premium becomes more volatile, and the agent with the least favorable

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\(^1\)See also David (1997), Cagetti, Hansen, Sargent, and Williams (2002), David and Veronesi (2002), and David (2008).
economic outlook requires a large risk premium for holding the risky asset.

Uncertainty and disagreement interact with each other to generate additional implications. When uncertainty is high and there is sufficient disagreement among agents, the agent with the least favorable economic outlook requires an even larger risk premium for holding the risky asset. However, without disagreement, uncertainty does not affect the risk premium, which leads us to conclude that disagreement is an important channel through which uncertainty commands a risk premium in the economy.

Our model explains several empirical regularities. First, it delivers an amplification mechanism through which the impact of disagreement and uncertainty on volatility and risk premia is stronger during recessions or booms. In these periods, disagreement leads to a higher risk premium and uncertainty causes large and persistent fluctuations in volatility. These features rationalize the high levels of volatility and large risk premia observed not only during recessions (Schwert, 1989; Patton and Timmermann, 2010; Barinov, 2014), but also during booms such as the Nasdaq bubble in the late 1990's (Pastor and Veronesi, 2006).

Second, our paper provides a theoretical foundation for GARCH. The persistence of volatility has been described extensively in the empirical literature, but there is a paucity of theoretical explanations. The explanation that we provide here is based on structural uncertainty, which arises endogenously from learning about growth persistence.

Finally, our work rationalizes the empirical findings in Carlin, Longstaff, and Matoba (2014) who analyze how model disagreement in the MBS market affects asset prices. The authors show that there is a risk premium associated with disagreement and that disagreement varies over time. It appears that disagreement rises during periods of large market movements, which is consistent with our idea of structural disagreement. Further, the authors show that disagreement is the primary channel through which uncertainty leads to trading volume. Ostensibly, this is consistent with our finding that uncertainty is only incorporated into risk premia when sufficient disagreement is present.

Our approach departs from the existing asset pricing literature in two ways. First, instead of assuming that agents learn about the unobserved growth rate of the economy, we propose a model whereby agents observe the growth rate but learn about its persistence. As we show in the paper, this type of learning is associated with time-varying uncertainty, which contrasts with learning

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2A few preference-based foundations for volatility clustering are provided by Campbell and Cochrane (1999), Barberis, Huang, and Santos (2001), and McQueen and Vorkink (2004). See also Osambela (2015).

3Timmermann (2001) shows that existence of structural breaks in the fundamental process induces volatility clustering. David (1997) and Veronesi (1999) show that, when the fundamental follows a Markov-switching process, learning implies volatility clustering. A commonality between our work and theirs is that agents have imperfect information about the economic model, although our model does not feature structural breaks or a finite number of states. See also Collin-Dufresne, Johannes, and Lochstoer (2015), who show that parameter learning generates long-lasting risks when a representative agent has a preference for early resolution of uncertainty.
models with similar structures\textsuperscript{4} and is empirically plausible (Jurado, Ludvigson, and Ng, 2015).\textsuperscript{5}

Second, we depart from the usual modeling approach to study the effect of heterogeneous beliefs on asset prices. Typically, this literature assumes that agents agree to disagree about the unobservable fundamental—the expected dividend growth—(e.g., Scheinkman and Xiong, 2003; Dumas, Kurshev, and Uppal, 2009).\textsuperscript{6} Instead, we assume that the fundamental is publicly observed and agreed upon at all times, but agents disagree about the parameters of the model that govern its dynamics. This setup is motivated by recent empirical evidence of model disagreement in MBS markets: Carlin et al. (2014) document substantial disagreement among Wall Street mortgage dealers about prepayment speed forecasts, although all of the dealers in the survey are large financial institutions having access to all publicly available information and only very little private information.\textsuperscript{7}

The rest of the paper proceeds as follows. Section 2 defines our model and the learning processes that the agents use. Section 3 characterizes the market equilibrium and shows how uncertainty and disagreement affect asset prices. Section 4 presents numerical results, with parameters resulting from an empirical calibration. Section 5 concludes. The Appendix contains all proofs, investigates the accuracy of our numerical approximation, and describes our calibration exercise.

\section{The Model}

In this section, we first describe the economy in which two agents need to learn about the persistence of the expected growth rate of aggregate consumption. In other words, agents are uncertain about the length of the business cycle and therefore need to estimate it. Then, we solve the learning problem of both agents and describe how their views differ.

Consider a pure exchange economy defined over a continuous-time horizon $[0, \infty)$, in which a single consumption good serves as the numéraire. There is a single risky asset (the stock) in positive unit supply, which is the claim to the aggregate consumption stream. There is also a risk-free asset.
available in zero net supply. The aggregate consumption/dividend stream $\delta$ follows the process

$$\frac{d\delta_t}{\delta_t} = \tilde{f}_t dt + \sigma_\delta dW^\delta_t$$

(1)

d$\tilde{f}_t$ = $\lambda_t \left( \bar{f} - \tilde{f}_t \right) dt + \sigma_f \left( \rho dW^\delta_t + \sqrt{1 - \rho^2} dW^f_t \right)$

(2)

d$\lambda_t$ = $\kappa (\bar{\lambda} - \lambda_t) dt + \Phi dW^\lambda_t$,

(3)

where $W^\delta, W^f, \text{and} W^\lambda$ are three independent Brownian motions under the physical (objective) probability measure $P$. The expected consumption growth rate $\tilde{f}$ in (2) is referred to as the fundamental, which mean-reverts to its long-term mean $\bar{f}$ at speed $\lambda$. The mean-reversion speed $\lambda$ is assumed to be mean-reverting process, as described in (3). The mean-reversion speed drives the length of recessions and expansions and can be influenced, for instance, by continuous technological change of the economic environment. The parameters $\sigma_\delta$ and $\sigma_f$ are the volatilities of the dividend growth and of the fundamental, and the parameters $\kappa$ and $\Phi$ govern the dynamics of the mean-reversion speed $\lambda$.

The economy is populated by two agents, $A$ and $B$. Agents trade with each other and derive utility from consumption. Each agent chooses a consumption-trading policy to maximize her expected lifetime utility

$$U_i = \mathbb{E}^i \left[ \int_0^\infty e^{-\beta t} \frac{c_{it}^{1-\alpha}}{1-\alpha} dt \right]$$

(4)

where $\beta > 0$ is the time discount rate, $\alpha > 0$ is the relative risk aversion coefficient, and $c_{it}$ denotes the consumption of agent $i \in \{A, B\}$ at time $t$. The expectation in (4) depends on agent $i$’s perception of future economic conditions and her preferences.

At all times, both agents observe the output $\delta$ and the fundamental $\tilde{f}$. We interpret the fundamental $\tilde{f}$ as the average/median forecast of the output growth rate among a large survey of professional forecasters. Agents in this economy observe this forecast and agree on it. The parameters $\bar{f}, \sigma_\delta, \sigma_f, \rho, \bar{\lambda}$, and $\Phi$ are commonly known. Given this, both agents are able to compute and observe $d\tilde{f}_t$ and $dW^\delta_t$. However, they are unable to observe the mean-reversion speed $\lambda$ and are therefore tasked with estimating it. This feature distinguishes our model from the previous literature (e.g., Scheinkman and Xiong, 2003; Dumas et al., 2009). Indeed, in other models, it is assumed that agents do not observe the fundamental $\tilde{f}$ and have heterogeneous beliefs about it. Here, both agents publicly observe $\tilde{f}$ and its evolution with no disagreement, but try to estimate its mean-reversion speed $\lambda$ in order to predict the length of the business cycle. That is, the question is not whether the economy is growing or not, but how long the boom or recession will last before the economy moves back to its known long-term growth rate $\bar{f}$. For instance, the issue during the

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8The mean-reversion speed can become negative because it is modeled with an Ornstein-Uhlenbeck process. The parameter estimation performed in Appendix A.4, however, implies that the probability of the mean-reversion speed turning negative is small, and therefore insures that the fundamental is stationary. The stationarity properties of an Ornstein-Uhlenbeck process with stochastic mean-reversion speed are discussed by Benth and Khedher (2013).
Great Recession of 2007-08 was not whether we were in recession—it was pretty clear we were—but how long it would last. As we will show below, this alternative way of modeling learning and disagreement leads to theoretical predictions distinct from the rest of the literature.

We introduce heterogeneity of beliefs by adopting the “difference-of-opinion” approach (see, e.g., Harris and Raviv, 1993; Kandel and Pearson, 1995, and many others cited in the Introduction). We let each agent receive a different signal about $\lambda$:

$$
\begin{align*}
  ds^A_t &= \phi dW^\lambda_t + \sqrt{1-\phi^2} dW^A_t, \\
  ds^B_t &= \phi dW^\lambda_t + \sqrt{1-\phi^2} dW^B_t,
\end{align*}
$$

where $W^A$ and $W^B$ are independent Brownian motions, also independent from $W^\delta$, $W^\lambda$, and $W^f$. Agents do not learn from each others’ behavior, i.e., they do not trust the information source of the other agent. The parameter $\phi$ defines the level of divergence of opinion: if $\phi = 1$, agents are in perfect agreement and the setup reduces to a representative agent economy; if $0 < \phi < 1$, the two signals are different and as a results agents will end up in disagreement about $\lambda$. If $\phi = 0$, neither of the signals is informative and therefore agents are again in perfect agreement.

### 2.1 Learning

The filtered mean-reversion speed $\hat{\lambda}$ and its posterior variance—uncertainty—$\gamma$ are such that $\lambda_t$ is normally distributed with mean $\hat{\lambda}_t$ and variance $\gamma_t$. Based on (1)-(6), the Kalman filter implies that agent $i \in \{A, B\}$ has the following system of state variables in mind:

$$
\begin{align*}
  d\zeta_t &= \left( \tilde{f}_t - \frac{1}{2} \sigma_\delta^2 \right) dt + \sigma_\delta d\tilde{W}^\delta_t \\
  df_t &= \hat{\lambda}_t \left( \tilde{f}_t - \hat{f}_t \right) dt + \sigma_f \rho d\tilde{W}^\delta_t + \sigma_f \sqrt{1-\rho^2} d\tilde{W}^{fi}_t \\
  d\lambda_t^i &= \kappa \left( \bar{\lambda} - \hat{\lambda}_t^i \right) dt + \frac{\gamma_t}{\sigma_f \sqrt{1-\rho^2}} \left( \tilde{f}_t - \hat{f}_t \right) d\tilde{W}^{fi}_t + \phi \Phi d\tilde{W}^{si}_t
\end{align*}
$$

where $\zeta \equiv \log \delta$ and $\tilde{W}^\delta$, $\tilde{W}^{fi}$, and $\tilde{W}^{si}$ are independent Brownian motions under the probability measure of agent $i$, as defined in Appendix A.1. Note that, if the parameter $\phi$ is positive but lower than one, agents come up with different estimates of $\lambda$ because of the last term in (9).

The dynamics of the filter $\hat{\lambda}$ are particular to this learning exercise. First, its instantaneous variance is directly driven by the fundamental and it is higher when the expected growth rate is away from its long-term mean. This is because agents are provided with accurate information

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9 Besides long-run behavior, there are other dimensions of parameter uncertainty analyzed in the literature, such as tail events (Liu, Pan, and Wang, 2005) or regime changes (Ju and Miao, 2012). See also Collin-Dufresne et al. (2015) and their Online Appendix for a good survey of the literature. The parameter $\lambda$, which is the focus of this paper, has spawned serious interest and disagreement among practitioners and academics. It has been at the center of understanding recoveries following economic crises (Reinhart and Rogoff, 2009; Howard, Martin, and Wilson, 2011; Bernanke, 2013) and in studying long-term trends in economic growth (Beeler and Campbell, 2012; Bansal, Kiku, and Yaron, 2012; Summers, 2014; Hamilton, Harris, Hatzis, and West, 2015).

10 See Theorem 12.7 in Liptser and Shiryaev (2001) and Appendix A.1 for details.
about the mean-reversion speed $\lambda$ when $f - \tilde{f}$ is large, as can be seen from (8). When $\tilde{f}_t - \tilde{f} = 0$, fluctuations in the fundamental do not provide any valuable information about the mean-reversion speed because these fluctuations are pure noise. Second, $\lambda$ can exhibit regimes of positive or negative correlation with the fundamental. When $\tilde{f}$ is high, positive $d\tilde{W}_{fi}$ shocks lower $\lambda$. For example, if today the economy is in good times (i.e., $\tilde{f}_t > \bar{f}$) and agents observe a positive $d\tilde{f}_t$, then $\lambda$ is likely to be small (i.e., the present boom is likely to persist). Following the same intuition, if the economy is in bad times (i.e., $\tilde{f}_t < \bar{f}$) and agents observe a negative $d\tilde{f}_t$, then $\lambda$ is likely to be small (i.e., the present recession is likely to persist). In other words, agents form extrapolative expectations: they regard unusual good or bad past performance of the economy as indicators of a slow-moving economy, or as the economy’s “new normal.”

The dynamics of uncertainty $\gamma$ are the same for both agents

$$
\frac{d\gamma_t}{dt} = (1 - \phi^2)\Phi^2 - 2\kappa\gamma_t - \frac{(\bar{f} - \tilde{f}_t)^2}{\sigma_f^2(1 - \rho^2)}\gamma_t^2.
$$

(10)

The last term in (10) distinguishes our setup from previous models. The deviation of the fundamental from its long-term mean affects the amount of uncertainty the agents have about $\lambda$. When $\tilde{f}_t - \tilde{f} \neq 0$, fluctuations in the fundamental are informative. On the other hand, if $\tilde{f}_t - \tilde{f} = 0$, fluctuations in the fundamental do not provide any valuable information. Therefore, fluctuations in the fundamental generate fluctuations in uncertainty, a particular feature which distinguishes learning about the persistence of the fundamental from learning about the fundamental itself. Indeed, learning about the fundamental implies that uncertainty converges rapidly to a constant steady-state, and hence does not influence the dynamics of asset prices (e.g., Dumas et al., 2009). Fluctuating uncertainty, however, also arises when agents learn about a Markov chain or about a regression coefficient (e.g., David, 1997; Veronesi, 2000; Xia, 2001).

The observable fundamental provides the link between the probability measures of the agents, $\mathbb{P}^A$ and $\mathbb{P}^B$:

$$
d\tilde{W}_{fi}^A = d\tilde{W}_{fi}^B + \frac{(\lambda_t^B - \lambda_t^A)(\bar{f} - \tilde{f}_t)}{\sigma_f \sqrt{1 - \rho^2}} dt,
$$

(11)

where $\lambda_t^B - \lambda_t^A$ is the difference in the way in which agents rationalize the fundamental. Since each agent perceives the economy under a different probability measure, the $\mathbb{P}^A$-expectation of any random variable $X$ can also be computed under $\mathbb{P}^B$ by using the following relation

$$
\mathbb{E}^A [X] = \mathbb{E}^B [\eta X],
$$

(12)

where the process $\eta$ is the change of measure from $\mathbb{P}^B$ to $\mathbb{P}^A$. 

7
The change of measure $\eta$ satisfies

$$\eta_t \equiv \frac{dP^A}{dP^B}\bigg|_{\mathcal{F}_t} = e^{-\frac{1}{2} \int_0^t \left( \frac{\hat{\lambda}_B^t - \hat{\lambda}_A^t}{\sigma f \sqrt{1 - \rho^2}} \right)^2 ds - \int_0^t \frac{\hat{\lambda}_B^t - \hat{\lambda}_A^t}{\sigma f \sqrt{1 - \rho^2}} d\hat{W}^B}_t, \quad (13)$$

where $\mathcal{F}_t$ is the observation filtration at time $t$ and $\eta$ has the following dynamics

$$\frac{d\eta_t}{\eta_t} = -\frac{\left( \hat{\lambda}_B^t - \hat{\lambda}_A^t \right) \left( \bar{\eta} - \tilde{\eta} \right)}{\sigma f \sqrt{1 - \rho^2}} d\hat{W}^B}_t. \quad (14)$$

Going forward, we define two important state variables. The first is the structural uncertainty

$$U \equiv (\bar{\eta} - \tilde{\eta})\gamma, \quad (15)$$

which is the product of uncertainty and the state of the economy. It appears in the dynamics of the mean-reversion speed (9) and uncertainty (10). The second is the structural disagreement

$$D \equiv (\bar{\eta} - \tilde{\eta})\left( \hat{\lambda}_B - \hat{\lambda}_A \right), \quad (16)$$

which is the product of the difference in agents’ estimates of $\lambda$ and the state of the economy. It drives the diffusion of the change of measure (14). Both $U$ and $D$ will play a key role in understanding our asset-pricing results.

3 Asset Prices

In this section, we first characterize the state-price density and describe the risk-free rate and market price of risk. We then solve for the equilibrium stock price, stock-return volatility, risk premium, and Sharpe ratio.

3.1 State-Price Density, Risk-Free Rate, and Market Price of Risk

In what follows, we choose to work under agent $B$’s probability measure $\mathbb{P}^B$.\footnote{Note that the results presented thereafter are qualitatively the same if we work under agent $A$’s measure.} Assuming that markets are complete, we follow Cox and Huang (1989) and solve for the equilibrium using the martingale approach. The state-price density perceived by agent $B$, $\xi^B$, satisfies

$$\xi^B_t = e^{-\beta t} \left[ \left( \frac{\eta_t}{\Lambda_A} \right)^{1/\alpha} + \left( \frac{1}{\Lambda_B} \right)^{1/\alpha} \right]^\alpha$$

$$= \frac{1}{\Lambda_B} e^{-\beta t} \delta_t^{1-\alpha} \omega_B^{-\alpha}, \quad (17)$$

$$= \frac{1}{\Lambda_B} e^{-\beta t} \delta_t^{1-\alpha} \omega_B^{-\alpha}, \quad (18)$$
where \( \Lambda_A \) and \( \Lambda_B \) are the Lagrange multipliers associated with the static budget constraints of agents \( A \) and \( B \), and \( \omega_{it} \equiv c_{it}/\delta_t \) the consumption share of agent \( i \in \{A,B\} \) at time \( t \).\(^{12}\) The consumption shares are functions of the change of measure \( \eta_t \) and their sum equals one.

According to (18), agent \( B \)’s state-price density depends on the aggregate consumption \( \delta \) but also on her consumption share \( \omega_{Bt} \). That is, the agent cares not only about the aggregate level of consumption (which would be the case in a representative agent economy), but also on how much of the aggregate output she shares with agent \( A \).

The following proposition provides the equilibrium risk-free rate, the market price of risk vector perceived by agent \( B \), and the price-dividend ratio for the risky asset.

**Proposition 1.** The risk-free rate and the market price of risk vector perceived by agent are:

\[
\begin{align*}
    r_t &= \beta + \alpha \tilde{f}_t - \frac{1}{2} \alpha (\alpha + 1) \sigma_\delta^2 + \frac{1}{2} \frac{\alpha - 1}{\alpha} \frac{D_t^2}{\sigma_\delta^2 (1 - \rho^2)} \omega_{At} \omega_{Bt} \\
    \theta^B_t &= \left( \alpha \sigma_\delta \frac{D_t}{\sigma_\delta \sqrt{1 - \rho^2}} \omega_{At} 0 0 \right)^\top.
\end{align*}
\]

Assuming that the coefficient of relative risk aversion \( \alpha \) is an integer,\(^{13}\) the equilibrium price-dividend ratio of the risky asset satisfies

\[
\frac{S_t}{\delta_t} = \sum_{j=0}^\alpha \left( \frac{\alpha}{j} \right) \omega_{At}^{\alpha-j} F_j(Z_t),
\]

where

\[
F_j(Z_t) = \mathbb{E}_t^B \left[ \int_t^\infty e^{-\beta (u-t)} \left( \frac{\eta_u}{\eta_t} \right)^\frac{j}{\delta u} \left( \frac{\delta u}{\delta_t} \right)^{1-\alpha} du \right].
\]\n
The 4-dimensional vector of state variables, \( Z \), is defined by \( Z \equiv (\tilde{f}, \hat{\lambda}^A, \hat{\lambda}^B, \gamma)^\top \).

**Proof.** See Appendices A.2 and A.3.

By inspection of (19) and (20), structural disagreement is a key driver of both the risk-free rate and the market price of risk. The price-dividend ratio expressed in Equation (21) consists in a weighted sum of expectations, with weights characterized by the consumption shares.

The first three components in (19) form the usual risk-free rate in a representative agent economy. First, the risk-free rate increases with the discount factor \( \beta \). Second, the risk-free rate increases with the fundamental \( \tilde{f} \). In this case, agents expect higher future consumption and hence lower future marginal utility. Future payments due to saving have lower value, which decreases

\(^{12}\)Derivations are provided in Appendix A.2.

\(^{13}\)This assumption simplifies the calculus. To the best of our knowledge, it has been first pointed out by Yan (2008) and Dumas et al. (2009). If the coefficient of relative risk aversion is real, the computations can still be performed using Newton’s generalized binomial theorem. Bhamra and Uppal (2013) offer a comprehensive analysis for all possible values of the risk aversion.
the demand for the risk-free asset and increases the equilibrium risk-free rate. Third, the risk-free rate decreases with the volatility of aggregate consumption \( \sigma_\delta \). The higher the volatility, the more agents demand risk-free payments and hence a lower risk-free rate is necessary to clear the market for borrowing and lending.

Finally, the last term in (19) comprises additional effects on the risk-free rate due to disagreement. When the risk aversion coefficient is larger than one, both agents in this economy expect a larger consumption share in the future under their own probability measure (see Equation (61) in Appendix A.2), and thus lower future marginal utility. This effect arises because each agent believes that the other agent’s model is partially inaccurate. Therefore, agents decide to save less today. This increases the equilibrium risk-free rate, effect further magnified if disagreement is large.

Given this, structural disagreement affects the risk-free rate (i) when \( \alpha \neq 1 \), (i.e., agents do not have logarithmic utility), (ii) when the fundamental deviates from its long-term mean \( \bar{f} \) (otherwise, \( D_t = 0 \)), and (iii) when there is a reasonable amount of parity among agents (i.e., it is not the case that \( \omega_i t >> \omega_j t \)).

Since there are four sources of information available to agents \( (d\delta, d\bar{f}, ds_A, \text{and} ds^B) \), the market price of risk perceived by agent \( B \) in (20) loads on four independent sources of risk: aggregate consumption risk \( \hat{W}^\delta \), fundamental risk \( \hat{W}^fB \), agent \( B \)'s information risk \( \hat{W}^sB \), and agent \( A \)'s information risk \( \hat{W}^sA^* \).

Only aggregate consumption risk and fundamental risk, however, are priced. First, agent \( B \) requires a positive price for bearing the risk of fluctuations in aggregate consumption \( \delta \), as determined by the first element in the vector (market price of aggregate consumption risk). Second, agent \( B \) requires a price for bearing the risk of fluctuations in her own consumption share (market price of consumption share risk). This second term comes through the effect of the revision of both agents’ forecasts of their relative consumption shares; it is related to the Brownian driving the shocks on the fundamental \( \hat{W}^fB \) because agents disagree about the interpretation of these shocks, as can be seen from the dynamics of the change of measure in Equation (14).

To understand the sign of the market price of consumption share risk, it is instructive to consider two situations. Notice first that, if \( D_t > 0 \), agent \( B \)'s model has a more favorable economic outlook. In this case, a positive shock to the fundamental increases agent \( B \)'s consumption share. This induces a positive correlation between her consumption share and the fundamental, and thus agent \( B \) requires a positive premium to bear consumption share risk. Alternatively, if \( D_t < 0 \), agent \( B \)'s model has a less favorable outlook. In this case, her consumption share is negatively correlated with fundamental risk, and agent \( B \) is willing to pay a price to bear consumption share risk.

The magnitude of the market price of consumption share risk depends on how much agent \( A \) consumes, \( \omega_A \). Naturally, agent \( B \) requires a higher market price of risk when the proportion of agent \( A \) in the economy is large. Furthermore, consumption shares do not fluctuate when \( f_i = \bar{f} \) or when \( \hat{\lambda}_t^A = \hat{\lambda}_t^B \) (in these cases \( D_t = 0 \) and agents have the same forecasts) and thus there is no

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14This arises when (i) the economy is going through good times \((\bar{f} - f_i < 0)\) and agent \( B \) believes the fundamental to be more persistent than agent \( A \) \((\hat{\lambda}_B - \hat{\lambda}_A < 0)\), or (ii) when the economy is going through bad times and agent \( B \) believes the fundamental to be less persistent than agent \( A \).
market price of consumption share risk.

3.2 Return Volatility and Risk Premia

Proposition 2 below characterizes the equilibrium stock-return volatility and risk premium.

**Proposition 2.** The state variables in this economy are $\tilde{f}_t$, $\tilde{\lambda}_A^t$, $\tilde{\lambda}_B^t$, $\gamma_t$, and $\mu_t \equiv \log \eta_t$. The diffusion vector of stock returns, $\Sigma$, and the risk premium perceived by agent $B$, $RP_B$, satisfy

$$\Sigma = \begin{pmatrix} \frac{S_f}{S} \sigma_f \sqrt{1 - \rho^2} + \frac{\sigma_\delta + \frac{S_f}{S} \sigma_{fp}}{\sigma_f \sqrt{1 - \rho^2}} \left[ -\frac{S_\mu}{S} \mathcal{D} + \left( \frac{S_{\lambda A}}{S} + \frac{S_{\lambda B}}{S} \right) \mathcal{U} \right] \\
\frac{S_{\lambda A}}{S} \phi \Phi \left( \frac{S_{\lambda A}}{S} \phi^2 + \frac{S_{\lambda B}}{S} \right) \\
\frac{S_{\lambda A}}{S} \phi \Phi \sqrt{1 - \phi^4} \end{pmatrix}^\top$$

(23)

$$RP_B \equiv \Sigma \theta^B$$

(24)

$$= \alpha \sigma_\delta \left( \sigma_\delta + \frac{S_f}{S} \sigma_{fp} \right) + \frac{S_f}{S} \mathcal{D} \omega_A + \left[ \left( \frac{S_{\lambda A}}{S} + \frac{S_{\lambda B}}{S} \right) \mathcal{U} \mathcal{D} - \frac{S_\mu}{S} \mathcal{D}^2 \right] \frac{\omega_A}{\alpha^2 (1 - \rho^2)}$$

(25)

where $|| \cdot ||$ is the norm operator, $S_y$ denotes the partial derivative of the stock price with respect to the state variable $y$, and $\mathcal{U}$ and $\mathcal{D}$ are defined in (15) and (16).

**Proof.** Application of Itô’s lemma on the stock price defined in (21).

Because the economy is driven by four sources of risk, the stock-return diffusion $\Sigma$, expressed in Equation (23), has four components. The instantaneous volatility of stock returns is computed as the norm of the diffusion vector, $\sigma \equiv ||\Sigma||$. However, as we describe in more details later, only the second component of $\Sigma$ significantly affects the volatility.\textsuperscript{15} The state variables that directly impact this component are the structural disagreement $\mathcal{D}$ and the structural uncertainty $\mathcal{U}$. We analyze the impact of these objects in Sections 4.1 and 4.2.

The risk premium perceived by agent $B$ in (25) is computed as the vector product between the market price of risk $\theta^B$ and the stock return diffusion $\Sigma$. Because the market price of risk is driven solely by structural disagreement, it follows that uncertainty affects the risk premium only when $\tilde{\lambda}_B \neq \tilde{\lambda}_A$: if we set $\mathcal{D} = 0$ in (25), then uncertainty has no direct impact on the risk premium. This implies that disagreement is one channel through which uncertainty impacts the risk premium. We will explore this more in Section 4.2.

Learning about the length of the business cycle connects both uncertainty and disagreement with the observable fundamental $\tilde{f}$, as shown in (15) and (16). This feature is particular to our model and implies that uncertainty and disagreement have an impact only when the economy is away from “normal times” (i.e., when $\tilde{f} \neq \tilde{f}$). As the fundamental is closer to its long-run mean, both structural uncertainty and structural disagreement get close to zero. Conversely, a

\textsuperscript{15}As we show numerically, the partial derivatives of the stock price with respect to the state variables exhibit little variation and therefore do not significantly drive the volatility.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility of dividend growth</td>
<td>$\sigma_{\delta}$</td>
<td>0.0132*** ( (2.95 \times 10^{-4}) )</td>
</tr>
<tr>
<td>Long-term mean of the fundamental</td>
<td>$\bar{f}$</td>
<td>0.0261*** ( (6.59 \times 10^{-4}) )</td>
</tr>
<tr>
<td>Volatility of fundamental</td>
<td>$\sigma_f$</td>
<td>0.0195*** ( (1.41 \times 10^{-3}) )</td>
</tr>
<tr>
<td>Correlation between dividend and fundamental</td>
<td>$\rho$</td>
<td>0.1501*** ( (2.16 \times 10^{-2}) )</td>
</tr>
<tr>
<td>Persistence of mean-reversion speed</td>
<td>$\kappa$</td>
<td>0.4989*** ( (0.1101) )</td>
</tr>
<tr>
<td>Long-term mean of mean-reversion speed</td>
<td>$\lambda$</td>
<td>1.1503*** ( (6.62 \times 10^{-2}) )</td>
</tr>
<tr>
<td>Volatility of mean-reversion speed</td>
<td>$\Phi$</td>
<td>0.6495*** ( (6.23 \times 10^{-2}) )</td>
</tr>
<tr>
<td>Difference-of-beliefs parameter</td>
<td>$\phi$</td>
<td>0.0955** ( (3.93 \times 10^{-2}) )</td>
</tr>
</tbody>
</table>

Table 1: Calibration to the U.S. economy (SMM estimation)

Parameter values resulting from a SMM estimation with 13 moment conditions. Standard errors are in brackets and statistical significance at the 10%, 5%, and 1% levels is labeled *, **, and ***, respectively.

fundamental far from its long-run mean enhances the effects of uncertainty and disagreement on asset returns.

Given the nature of the expressions in (23)-(25), further characterizing the effects of uncertainty and disagreement necessitates a numerical implementation. In the analysis that follows, we set $\alpha = 3$, $\beta = 0.01$, and the ratio of Lagrange multipliers equal to one. This last choice assures that both agents are endowed with the same initial share of consumption. Then, we calibrate the model to the real U.S. GDP growth rate and its 1-quarter-ahead median analyst forecast using the Simulated Method of Moments.\footnote{See Appendix A.4 for further details on the estimation method.}

Our parameter estimates and their statistical significance are summarized in Table 1. All but the disagreement parameter $\phi$ are statistically significant at the 99% confidence level. The volatility of dividend growth $\sigma_{\delta}$, the long-term mean of the fundamental $\bar{f}$, and the volatility of the fundamental $\sigma_f$ are consistent with the values used in the asset-pricing literature (e.g., Bansal and Yaron, 2004; Brennan and Xia, 2001; Bansal, Kiku, and Yaron, 2010; Croce, Lettau, and Ludvigson, 2014). The positive correlation $\rho$ between the fundamental and the dividend growth rate reflects the fact that analysts use the observation of the GDP to forecast its future growth: analysts adjust their forecasts upward when they observe a positive growth surprise, and downward when they observe a negative growth surprise. Moreover, a correlation coefficient smaller than one shows that, on top of the observation of the GDP, analysts use other sources of information to infer the expected growth. The long-term mean-reversion speed of the fundamental implies a half-life of approximately seven
months, which is significantly shorter than what is assumed in the long-run risk literature. The mean-reversion speed itself is relatively persistent as its half-life is approximately 17 months. It is also highly volatile (the estimated parameter $\Phi$ is large), which lends support to our assumption of time-varying mean-reversion speed.

4 Results

In this section, we first show that the stock-return volatility is a U-shaped function of the structural uncertainty. Since the structural uncertainty is linear in the fundamental, volatility is low in normal times and is high during recessions and booms. Then, we show that the risk premium decreases with structural disagreement, and that the impact of uncertainty on the risk premium is weak. That is, the relation between volatility and the risk premium is ambiguous.

4.1 Stock-Return Volatility

Equation (23) of Proposition 2 outlines a non-linear relation between stock-return volatility $\sigma$ and the fundamental $\tilde{f}$ ceteris paribus. To see this, we plot in the left panel of Figure 1 the stock-return volatility versus the fundamental. When $\gamma = 0$, the agents learn perfectly about $\lambda$ and volatility does not depend on the fundamental. In this case, the structural uncertainty $\mathcal{U}$ is also zero. When uncertainty increases, volatility is higher when the fundamental is far away from its long-term mean. In this case, the effect of uncertainty on volatility is amplified by the state of the economy.

With a few exceptions (David, 1997; Veronesi, 1999; Xia, 2001), models of learning do not generate fluctuations in uncertainty. In our setup, these fluctuations arise endogenously from the learning exercise: when agents try to estimate the length of the business cycle, uncertainty depends on the fundamental and thus fluctuates with the state of the economy. This, in equilibrium, generates fluctuations in volatility.

Based on (23), it appears that stock-return volatility also depends on $\tilde{\lambda}_B - \tilde{\lambda}_A$. However, it turns out that this relationship is significantly weaker. In the right panel of Figure 1, we plot the stock-return volatility versus the fundamental for different values of disagreement. The plot shows that disagreement has only a marginal effect on volatility.

We confirm the results from Figure 1 by simulating the economy 10,000 times at a 1-week frequency over 50 years. For each simulated point, we compute the structural uncertainty $\mathcal{U}$ and plot it against the volatility of stock returns. We then perform a quadratic fit of volatility on structural uncertainty. The quadratic fit line is shown in Figure 2 (solid line), where we also plot the 95% confidence intervals across simulations. The narrow confidence intervals suggest that

\footnote{A common empirical observation is that bad economic times tend to be characterized by higher uncertainty. Although this is not a feature of our model, the results from Figure 1 suggest that we should observe counter-cyclical volatility, i.e., higher volatility during times of higher uncertainty. Veronesi (1999) obtains a similar result in a single-agent economy in which the set of possible drifts for the aggregate dividend is finite. This discreteness of the state space generates fluctuating uncertainty.}

13
Figure 1: Volatility vs. fundamental
The left panel depicts the stock-return volatility, $\sigma$, against the fundamental $\tilde{f}$ for different values of the uncertainty $\gamma$. For this plot, structural disagreement is fixed to zero. The right panel depicts stock-return volatility, $\sigma$, against the fundamental $\tilde{f}$ for different values of structural disagreement $\Delta \hat{\lambda} \equiv \hat{\lambda}^B - \hat{\lambda}^A$. For this plot, uncertainty is fixed at its steady state value, $\gamma_{ss} = 0.42$, defined in Appendix A.3. The dashed vertical lines mark the steady state level of the fundamental, $\bar{f}$. Parameter values are provided in Table 1.

Figure 2: Volatility vs. structural uncertainty: quadratic fit on simulated data
The plot depicts the stock-return volatility, $\sigma$, against the structural uncertainty $\mathcal{U}$ defined by $(\tilde{f} - \hat{f})\gamma$. The solid line represents a quadratic fit on the model-generated data resulted from 10,000 simulations of the economy at weekly frequency over 50 years. The dashed lines represent the 95% confidence bands. Parameter values are provided in Table 1. Initial values are $\tilde{f}_0 = \bar{f}$, $\hat{\lambda}^A_0 = \hat{\lambda}^B_0 = \bar{\lambda}$, $\delta_0 = \eta_0 = 1$, and $\gamma_0 = \gamma_{ss}$, where $\gamma_{ss}$ is defined in Appendix A.3.

structural uncertainty generates most of the fluctuations in volatility, whereas disagreement has only a minor effect, in line with our interpretation of Figure 1.
According to Figure 2, fluctuations in the structural uncertainty $U$ are the main driver of fluctuations in volatility. This implies that the persistence of stock-return volatility is directly related to the persistence of uncertainty. Given that uncertainty is endogenously generated by the learning of agents, it is a strongly persistent process\textsuperscript{18} and consequently the structural uncertainty itself becomes persistent.\textsuperscript{19} Stock-return volatility is thus persistent not because some exogenous variable is assumed to be persistent, but because the uncertainty resulting from learning about the mean-reverting parameter $\lambda$ is persistent. Learning about persistence offers thus a theoretical foundation of the GARCH behavior commonly observed in financial markets (Engle, 1982; Bollerslev, 1986).

### 4.2 Equity Risk Premium

Proposition 2 shows that the structural uncertainty $U$ and the structural disagreement $D$ have a direct impact on the risk premium perceived by agent $B$. The equity risk premium is defined as the dot product between the market price of risk and the stock return diffusion. This has an important implication: because structural disagreement is the sole driver of the market price of risk, uncertainty affects the risk premium only when disagreement is present. In other words, in this economy, disagreement is the primary channel through which uncertainty affects the risk premium.

To illustrate this effect, we plot in the left panel of Figure 3 the risk premium as a function of the fundamental $\tilde{f}$ for different values of disagreement. The plot shows that agent $B$ requires a high premium to hold the risky asset when her model forecasts a less favorable economic outlook, i.e., when $D < 0$. In the right panel, we consider the case $\lambda^B - \lambda^A = -0.4$ and vary the amount of uncertainty (the dotted lines in the two panels are identical). The figure shows that uncertainty has an impact on the risk premium only when disagreement is different from zero. Specifically, higher uncertainty amplifies the risk premium required by agent $B$ to hold the asset (when agent $B$’s model forecasts a less favorable economic outlook).

According to the above discussion, the risk premium is mainly driven by the structural disagreement $D$. Furthermore, uncertainty magnifies the risk premium only when disagreement is different from zero. We confirm these effects again through simulations. We perform 10,000 simulations of the economy at a 1-week frequency over 50 years. For each simulated point, we compute the structural disagreement $D$ and plot it against the risk premium. We then perform a cubic fit of the risk premium on the structural disagreement. The cubic fit line is shown in Figure 4, where we also plot the 95% confidence intervals across simulations. The narrow confidence intervals sug-
suggest that indeed the relationship is non-linear and uncertainty affects the risk premium only when disagreement is away from zero. When the structural disagreement $\mathcal{D}$ is negative (resp. positive), agent $B$’s model has a less (resp. more) favorable economic outlook and therefore requires a high (resp. low) risk premium. This explains why the lines in Figure 4 have similar shapes as those in the right panel of Figure 3.

Disagreement about the length of the business cycle implies that the risk premium is, as the stock-return volatility, persistent in equilibrium. Indeed, the 1-week autocorrelation of the risk premium computed from our simulated sample is 0.948. Unlike volatility which is persistent because it is driven by uncertainty, the risk premium is persistent in our model because it is driven by a persistent disagreement; disagreement features a 1-week autocorrelation of 0.992.

4.3 Illustration: Asset Price Dynamics after a Growth Shock

We conclude this section with an illustration of the dynamics of the volatility and risk premium after the economy experiences a recessionary shock. Suppose that professional forecasters estimate the growth rate of the economy at -2.5%. Assume further that the two agents consider two different mean-reversion speeds, $\hat{\lambda}^A = 1.15$ and $\hat{\lambda}^B = 0.9$ (agent $B$’s model has a less favorable economic outlook—she expects a relatively longer recession).

Figure 5 illustrates the evolution of the volatility $\sigma$, risk premium $RP_B$, and disagreement $\hat{\lambda}^B - \hat{\lambda}^A$ over three years at weekly frequency. The solid lines represent averages across 10,000 simulations.\footnote{The average volatility across simulations and time is 4.5%, the average risk premium is 0.7%, and the average}
Figure 4: Volatility vs. structural disagreement: cubic fit on simulated data

The graph depicts the risk premium against the structural disagreement $D$ defined by $(\hat{\lambda}^B - \hat{\lambda}^A)(\tilde{f} - \bar{f})$. The solid line represents the cubic fit function on the model-generated data resulted from 10,000 simulations of the economy at weekly frequency over 50 years. The dashed lines represent the 95% confidence bands. Parameter values are provided in Table 1. Initial values are $\tilde{f}_0 = \bar{f}$, $\hat{\lambda}_0^A = \hat{\lambda}_0^B = \bar{\lambda}$, $\delta_0 = \eta_0 = 1$, and $\gamma_0 = \gamma_{ss}$, where $\gamma_{ss}$ is defined in Appendix A.3.

Panel (a) shows that volatility increases above 10% (in yearly terms) and then goes down slowly risk-free rate is 8.7%. These averages would better match their empirical counterparts if agents had habit formation, as in Chan and Kogan (2002), Xiouros and Zapatero (2010), Bhamra and Uppal (2013), and Ehling et al. (2013) among others. As this would not add any insights to the main predictions of our model, we decide to keep the setup simple and focus on the dynamic properties of asset prices.
over the next year (solid line)\textsuperscript{21}. We plot with dashed lines the 5th and 95th percentiles computed across 10,000 simulations. The take-away message of this exercise is that volatility is persistent and features a lot of variability. We have tried several other specifications, including an initial situation without disagreement, and the results are similar. This illustrates that learning about the mean-reversion speed of the fundamental induces GARCH-like variation in the volatility of stock returns.

Turning to panel (b), the risk premium required by agent $B$ for holding the asset is large and positive (recall that agent $B$’s model has a less favorable economic outlook). One important consideration here is that if we start with an initial situation without disagreement, then the risk premium is close to zero (although it still experiences fluctuations). Disagreement is therefore the main driver of risk premia, whereas uncertainty mostly explains the persistent fluctuations in volatility.

Finally, panel (c) shows the evolution of disagreement (i.e., the difference $\hat{\lambda}^B - \hat{\lambda}^A$) over the 3-year period of our simulated sample, averaged across 10,000 simulations. Disagreement takes a long time to converge back to zero.\textsuperscript{22} Furthermore, the 5th and 95th percentile lines show that disagreement is very volatile. Note also that, although agent $B$ believes initially that the fundamental is more persistent than agent $A$, there is a high chance than in the near future these beliefs will reverse and agent $B$’s model features a less persistent fundamental.

5 Conclusion

Understanding what drives stock market volatility remains one of the most important issues in finance. In this paper, we show that learning about the length of the business cycle generates many of the salient properties of volatility dynamics. When agents need to update their beliefs about the length of the business cycle, uncertainty about this length fluctuates, leading to new asset pricing implications. We characterize these effects in the paper and reach three primary conclusions. First, this type of learning implies persistent stock-return volatility, which provides a theoretical explanation for GARCH-type processes. Second, the state of the economy governs the “structural uncertainty”, which fluctuates and magnifies stock-market volatility in recessions and booms. Third, disagreement is associated with a risk premium and is a channel through which uncertainty commands a risk premium. To our knowledge, these are novel implications that add to the established literature on learning and heterogeneous beliefs and that provide plausible explanations for known empirical observations.

\textsuperscript{21}Given our parameter values, this increase in volatility is substantial: the volatility of the dividend growth is 1.3%, the volatility of the fundamental is 1.9% (see Table 1), and the risk aversion is $\alpha = 3$.

\textsuperscript{22}The stochastic process for disagreement is given in Equation (49) in the Appendix. Disagreement mean-reverts towards zero with a mean-reverting speed higher or equal to $\kappa$. It has a conditional variance of $2\Phi^2\phi^2(1-\phi^2)$ (see Equation (50) in the Appendix).
A Appendix

A.1 Filtering Problem

Following the notations of Liptser and Shiryaev (2001), agent $i$ observes the vector

$$
\begin{pmatrix}
\frac{d\zeta_t}{ds_t} \\
\frac{df_t}{ds_t}
\end{pmatrix} = (A_0 + A_1 \lambda_t) dt + B_1 dW_t^\lambda + B_2 \begin{pmatrix}
\frac{dW_t^\delta}{ds_t} \\
\frac{dW_t^f}{ds_t}
\end{pmatrix}
$$

(26)

where $i \in \{A, B\}$. The unobservable process $\lambda$ satisfies

$$
d\lambda_t = (a_0 + a_1 \lambda_t) dt + b_1 dW_t^\lambda + b_2 \begin{pmatrix}
\frac{dW_t^\delta}{ds_t} \\
\frac{dW_t^f}{ds_t}
\end{pmatrix}
$$

(28)

Therefore,

$$
\begin{align*}
bob &= b_1 b'_1 + b_2 b'_2 = \Phi^2 \\
BoB &= B_1 B'_1 + B_2 B'_2 = \begin{pmatrix}
\sigma_\delta^2 & \rho \sigma_f \sigma_\delta & 0 \\
\rho \sigma_f \sigma_\delta & \sigma_f^2 & 0 \\
0 & 0 & 1
\end{pmatrix} \\
boB &= b_1 B'_1 + b_2 B'_2 = \begin{pmatrix}
0 & 0 & \phi \Phi
\end{pmatrix}.
\end{align*}
$$

(30)

The estimated process defined by $\hat{\lambda}_t^i = \mathbb{E}^\mathcal{F}_t (\lambda_t | \mathcal{F}_t)$ has dynamics

$$
d\hat{\lambda}_t = \left(a_0 + a_1 \hat{\lambda}_t\right) dt + (bob + \gamma_t A'_i)(BoB)^{-1} \begin{pmatrix}
\frac{d\zeta_t}{ds_t} \\
\frac{df_t}{ds_t}
\end{pmatrix} - \begin{pmatrix}
A_0 + A_1 \hat{\lambda}_t
\end{pmatrix} dt
$$

(33)

where the uncertainty $\gamma$ solves the following Ordinary Differential Equation

$$
d\gamma_t = a_1 \gamma_t + \gamma_t a'_1 + bob - (bob + \gamma_t A'_i)(BoB)^{-1}(bob + \gamma_t A'_i)'.
$$

(34)

Consequently,

$$
d\hat{\lambda}_t = \kappa (\bar{\lambda} - \hat{\lambda}_t) dt + \begin{pmatrix}
0 \\
\frac{(f - \bar{f}) \gamma_t}{\sigma_f \sqrt{1 - \rho^2}} \\
\phi \Phi
\end{pmatrix} \begin{pmatrix}
\frac{dW_t^\delta}{ds_t} \\
\frac{dW_t^f}{ds_t} \\
\frac{dW_t^i}{ds_t}
\end{pmatrix}.
$$

(35)
where the three Brownians are independent and are defined as follows:

\[
d\hat{W}_t^\delta = dW_t^\delta \\
d\hat{W}_t^{fi} = \frac{1}{\sigma_f \sqrt{1 - \rho^2}} \left[ df_t - \lambda_i^t (\bar{f} - \hat{f}_t) dt - \sigma_f \rho dW_t^\delta \right] \\
d\hat{W}_t^{si} = ds_t^i
\]  

(36)  (37)  (38)

The dynamics of uncertainty are

\[
d\gamma_t = (1 - \phi^2) \Phi^2 - 2 \kappa \gamma_t - \frac{(\bar{f} - \hat{f}_t)^2}{\sigma_f^2 (1 - \rho^2)} \gamma_t^2.
\]  

(39)

A.2 Proof of Proposition 1

Consider the following 4-dimensional Brownian motion defined under the probability measure of agent \(B\):

\[
d\hat{W} = \begin{pmatrix} \hat{W}_t^\delta \\ \hat{W}_t^{fB} \\ \hat{W}_t^{sB} \\ \hat{W}_t^{sA*} \end{pmatrix}
\]

(40)

where the first three Brownians are defined in (36), (37) and (38). The last Brownian is defined such that (this ensures that the correlation between \(d\hat{W}_t^{sA}\) and \(d\hat{W}_t^{sA*}\) is \(\phi^2\)):

\[
d\hat{W}_t^{sA} = \phi^2 d\hat{W}_t^{sB} + \sqrt{1 - \phi^2} d\hat{W}_t^{sA*}
\]

(41)

Write the dynamics of the vector of state variables under the probability measure of agent \(B\):

\[
d\zeta_t = \left( \hat{f}_t - \frac{1}{2} \sigma_\delta^2 \right) dt + (\sigma_\delta \ 0 \ 0 \ 0) d\hat{W}_t \\
d\hat{f}_t = \lambda_t^B \left( \bar{f} - \hat{f}_t \right) dt + (\sigma_f \rho \ \sigma_f \sqrt{1 - \rho^2} \ 0 \ 0) d\hat{W}_t^{fi} \\
d\lambda_t^A = \left( \kappa \left( \lambda - \lambda_t^A \right) + \frac{\gamma_t}{\sigma_f^2 (1 - \rho^2)} \left( \bar{f} - \hat{f}_t \right)^2 \right) dt \\
+ \left( 0 \ \frac{\gamma_t}{\sigma_f \sqrt{1 - \rho^2}} \left( \bar{f} - \hat{f}_t \right) \phi \Phi \ 0 \right) d\hat{W}_t^{si} \\
d\lambda_t^B = \kappa \left( \lambda - \lambda_t^B \right) dt + \left( 0 \ \frac{\gamma_t}{\sigma_f \sqrt{1 - \rho^2}} \left( \bar{f} - \hat{f}_t \right) \phi \Phi \ 0 \right) d\hat{W}_t^{si} \\
d\gamma_t = \left( (1 - \phi^2) \Phi - 2 \kappa \gamma_t - \frac{\left( \bar{f} - \hat{f}_t \right)^2}{\sigma_f^2 (1 - \rho^2)} \gamma_t^2 \right) dt \\
d\mu_t = -\frac{1}{2} \left( \lambda_t^B - \lambda_t^A \right)^2 \left( \bar{f} - \hat{f}_t \right)^2 dt + \left( 0 \ -\frac{(\lambda_t^B - \lambda_t^A)(\bar{f} - \hat{f}_t)}{\sigma_f \sqrt{1 - \rho^2}} \ 0 \ 0 \right) d\hat{W}_t,
\]

(42)  (43)  (44)  (45)  (46)  (47)  (48)

The dynamics of disagreement are given by:

\[
d\left( \lambda_t^B - \lambda_t^A \right) = -\left( \kappa + \frac{\gamma_t}{\sigma_f^2 (1 - \rho^2)} \right) \left( \lambda_t^B - \lambda_t^A \right) dt + \left( 0 \ \phi \Phi (1 - \phi^2) \ \phi \Phi \sqrt{1 - \phi^2} \right) d\hat{W}_t,
\]

(49)
and thus its conditional variance is
\[ 2\phi^2 \Phi^2 (1 - \phi^2). \] (50)

The optimization problem of agent B is
\[
\begin{align*}
\max_{c_{Bt}} & \quad E \left[ \int_0^\infty e^{-\beta t} \frac{c_{Bt}^{1-\alpha}}{1-\alpha} dt \right] \\
\text{s.t.} & \quad E \left[ \int_0^\infty \xi_{Bt} c_{Bt} dt \right] \leq x_{B0},
\end{align*}
\] (51)

where \( \xi^B \) denotes the state-price density perceived by agent B and \( x_{B0} \) is her initial wealth. Under the probability measure \( P^B \), the problem of agent A is
\[
\begin{align*}
\max_{c_{At}} & \quad E \left[ \int_0^\infty e^{-\beta t} \frac{c_{At}^{1-\alpha}}{1-\alpha} dt \right] \\
\text{s.t.} & \quad E \left[ \int_0^\infty \xi_{Bt} c_{At} dt \right] \leq x_{A0}.
\end{align*}
\] (52)

Note that the change of measure enters directly the objective function of agent A but not its budget constraint (54). The reason is that the budget constraint depends on the state-price density perceived by agent B.

The first-order conditions are
\[
\begin{align*}
c_{Bt} &= \left( \Lambda_B e^{\beta t} \xi_t \right)^{-\frac{1}{\alpha}} \xi^B, \quad (55) \\
c_{At} &= \left( \frac{\Lambda_A}{\eta_t} e^{\beta t} \xi_t \right)^{-\frac{1}{\alpha}}, \quad (56)
\end{align*}
\]

where \( \Lambda_A \) and \( \Lambda_B \) are the Lagrange multipliers associated with the budget constraints of agents A and B, respectively. Summing up agents’ optimal consumption policies and imposing market clearing, i.e., \( c_{At} + c_{Bt} = \delta_t \), yields the state-price density perceived by agent B
\[
\xi_t^B = e^{-\beta t} \delta_t \left[ \left( \frac{\eta_t}{\Lambda_A} \right)^{1/\alpha} + \left( \frac{1}{\Lambda_B} \right)^{1/\alpha} \right]^\alpha. \quad (57)
\]

Substituting the state-price density \( \xi^B \) in the optimal consumption policies yields the following consumption sharing rules
\[
\begin{align*}
c_{At} &= \omega_{At} \delta_t, \\
c_{Bt} &= \omega_{Bt} \delta_t = (1 - \omega_{At}) \delta_t,
\end{align*}
\] (58) (59)

where \( \omega_{it} \) denotes agent \( i \)'s share of consumption at time \( t \) for \( i \in \{A, B\} \). Agent A’s share of consumption satisfies
\[
\omega_{At} = \frac{\left( \frac{\eta_t}{\Lambda_A} \right)^{1/\alpha}}{\left( \frac{\eta_t}{\Lambda_A} \right)^{1/\alpha} + \left( \frac{1}{\Lambda_B} \right)^{1/\alpha}}. \quad (60)
\]

The evolution of agent B’s consumption share under her own probability measure follows
\[
\frac{d\omega_{Bt}}{\omega_{Bt}} = \frac{1}{2} \frac{(1 - \omega_{Bt})(1 + \alpha - 2\omega_{Bt})}{\alpha^2 \sigma_f^2 (1 - \rho^2)} D_t^2 dt + \frac{1 - \omega_{Bt}}{\alpha \sigma_f \sqrt{1 - \rho^2}} D_t d\hat{W}_t^B. \quad (61)
\]
Notice that the drift is always positive (as long as there is disagreement). This is also the case for the consumption share of agent $A$, under her own probability measure.

As in Yan (2008) and Dumas et al. (2009), we assume that the coefficient of relative risk aversion $\alpha$ is an integer. In this case, the state-price density at time $u$ satisfies

$$
\xi^B_u = e^{-\beta u} \delta_u^{-\alpha} \left[ \left( \frac{1}{\Lambda_B} \right)^{1/\alpha} + \left( \frac{\eta_u}{\Lambda_A} \right)^{1/\alpha} \right]^\alpha \tag{62}
$$

$$
= e^{-\beta u} \delta_u^{-\alpha} \frac{1}{\Lambda_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left( \frac{\eta_u \Lambda_B}{\Lambda_A} \right)^{\frac{j}{\alpha}} \tag{63}
$$

$$
= e^{-\beta u} \delta_u^{-\alpha} \frac{1}{\Lambda_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left( \frac{1}{\eta_t} \right)^{\frac{j}{\alpha}} \left( \frac{\eta_u \Lambda_B}{\Lambda_A} \right)^{\frac{j}{\alpha}} \tag{64}
$$

$$
= e^{-\beta u} \delta_u^{-\alpha} \frac{1}{\Lambda_B} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left( \frac{\eta_u}{\eta_t} \right)^{\frac{j}{\alpha}} \left( \frac{\omega_A t}{1 - \omega_A t} \right)^j, \tag{65}
$$

where the last equality comes from the fact that

$$
\omega_A t = \left( \frac{\eta_u \Lambda_B}{\Lambda_A} \right)^{1/\alpha} \left( \frac{1}{\Lambda_B} \right)^{1/\alpha} + \left( \frac{\eta_t}{\Lambda_A} \right)^{1/\alpha}, \tag{66}
$$

$$
1 - \omega_A t = \left( \frac{1}{\Lambda_B} \right)^{1/\alpha} + \left( \frac{\eta_t}{\Lambda_A} \right)^{1/\alpha}, \tag{67}
$$

and consequently

$$
\left( \frac{\eta_u \Lambda_B}{\Lambda_A} \right)^{\frac{1}{\alpha}} = \frac{\omega_A t}{1 - \omega_A t}. \tag{68}
$$

Rewriting Equation (67) yields

$$
\left( \frac{1}{\Lambda_B} \right)^{1/\alpha} + \left( \frac{\eta_u}{\Lambda_A} \right)^{1/\alpha} = \left( \frac{1}{1 - \omega_A t} \right) \left( \frac{1}{\Lambda_B} \right)^{\frac{1}{\alpha}}, \tag{69}
$$

and thus

$$
\xi^B_t = e^{-\beta t} \delta_t^{-\alpha} \left( \frac{1}{1 - \omega_A t} \right)^{\alpha} \frac{1}{\Lambda_B} = \frac{1}{\Lambda_B} e^{-\beta t} (\delta_t \omega_B t)^{-\alpha}, \tag{70}
$$

which is Equation (18) in the text. Thus, the dynamics for the stochastic discount follow:

$$
\frac{d\xi_t}{\xi_t} = - \left[ \beta + \alpha \frac{\hat{f}_t}{\Lambda_B} - \frac{1}{2} \alpha (\alpha + 1) \sigma_a^2 \right] dt + \alpha^2 \frac{(1 - \omega_B t)(1 - \alpha)}{\alpha^2 \sigma_f^2 (1 - \rho^2)} D_t^2 dt - \frac{1}{2} \alpha (\alpha + 1) (1 - \omega_B t)^2 \left( \frac{\omega_A}{1 - \omega_A t} \right)^{\alpha} D_t^2 dt \tag{71}
$$

$$
+ \frac{\alpha}{2} \frac{\sigma_a}{\sigma_f \sqrt{1 - \rho^2}} \omega_A t \begin{pmatrix}
\frac{dW^t}{\hat{f}_B} \\
\frac{dW^t}{\Lambda_B} \\
\frac{dW^t}{\Lambda_A}
\end{pmatrix}, \tag{72}
$$

$$
\begin{pmatrix}
\alpha \sigma_a \\
\frac{\sigma_a}{\sigma_f \sqrt{1 - \rho^2}} \omega_A t
\end{pmatrix}
\begin{pmatrix}
\frac{dW^t}{\hat{f}_B} \\
\frac{dW^t}{\Lambda_B} \\
\frac{dW^t}{\Lambda_A}
\end{pmatrix}, \tag{73}
$$

22
We have

\[ r_t = \beta + \alpha f_t - \frac{1}{2} \alpha (\alpha + 1) \sigma^2_t + \frac{1}{2} \frac{\alpha - 1}{\sigma^2_f (1 - \rho^2)} \omega_A t \omega_B t. \]  

(74)

The price-dividend ratio satisfies

\[
\frac{S_t}{\delta_t} = E_t \left( \int_t^\infty \frac{\xi^B}{\xi^B} \frac{\delta u}{\delta t} \right)
\]

(75)

 \[= E_t \left( \int_t^\infty \frac{e^{-\beta u} \delta - \alpha}{\Lambda u} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \left( \frac{\eta_u}{\eta_t} \right)^j \left( \frac{1}{1 - \omega_A t} \right)^j \right) \]

(76)

\[= \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega_A (1 - \omega_A t)^{\alpha - j} E_t \left( \int_t^\infty \frac{e^{-\beta (u - t)} \left( \frac{\eta_u}{\eta_t} \right)^j \left( \frac{\delta_u}{\delta t} \right)^{1 - \alpha} \right) \right) \]

(77)

Let us define the function \( F_j(Z) \) as follows

\[ F_j(Z_t) = E_t \left[ \int_t^\infty e^{-\beta(u - t)} \left( \frac{\eta_u}{\eta_t} \right)^j \left( \frac{\delta_u}{\delta t} \right)^{1 - \alpha} \right] \]

(79)

where \( \epsilon = \frac{\alpha}{\alpha} \), \( j = 0, \ldots, \alpha \), \( \chi = 1 - \alpha \), and \( Z = (f, \tilde{\lambda}^A, \tilde{\lambda}^B, \gamma)^T \) is a vector of state-variables that does not comprise \( \zeta = \log \delta \) and \( \mu = \log \eta \).

Using these notations, the price-dividend ratio satisfies

\[ \frac{S_t}{\delta_t} = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega_A (1 - \omega_A t)^{\alpha - j} F_j(Z_t) \]

(80)

\[= \sum_{j=0}^{\alpha} \binom{\alpha}{j} \omega_A (1 - \omega_A t)^{\alpha - j} F_j(Z_t), \]

(81)

which is Equation (21) in Proposition 1.

### A.3 Exponential-Quadratic Approximation

We have

\[
\begin{align*}
\left( \frac{\eta_u}{\eta_t} \right)^\epsilon &= e^{-\frac{1}{2} f_t^e \left( \frac{\tilde{\lambda}^B - \tilde{\lambda}^A (f - f_t)}{\sigma f_j \sqrt{1 - \rho^2}} \right)^2 ds - f_t^e} \left( \begin{array}{c} 0 \\ e \left( \frac{\tilde{\lambda}^B - \tilde{\lambda}^A (f - f_t)}{\sigma f_j \sqrt{1 - \rho^2}} \right) \\ 0 \\ 0 \\ 0 \end{array} \right) d\tilde{W}_t, \\
\left( \frac{\delta_u}{\delta t} \right)^\chi &= e^{f_t^e \chi (f_t - \frac{1}{2} \sigma^2_t) ds + f_t^e} \left( \begin{array}{c} 0 \\ 0 \\ \chi \sigma \delta \\
0 \\ 0 \\ 0 \end{array} \right) d\tilde{W}_t,
\end{align*}
\]

(82)

(83)
where $\epsilon$ and $\chi$ are some constants. Therefore,

$$
\left( \frac{\eta_t}{\eta_t} \right) ^{\chi} = e^{f_0 ^{t} \left[ (\lambda^2 - \lambda^4) \frac{(f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right] + \frac{1}{2} \left[ \chi^2 \sigma^2 + \epsilon^2 \left( \frac{(\lambda^2 - \lambda^4) (f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right)^2 \right]} ds
$$

where $\bar{L}$ of the function $L$ is defined as follows:

$$
F = \frac{\lambda^2 - \lambda^4}{\sigma_f \sqrt{1 - \rho^2}} \frac{(f - f_J)}{\sigma_f \sqrt{1 - \rho^2}}
$$

where $\epsilon$ transforms this expression to obtain a $\bar{L}$ when defining the function $F$. Importantly, the second row defines a change of measure. The change of measure is

$$
\frac{d\hat{P}}{d\bar{P}} \bigg| _{t_t} \equiv \nu_t = e^{- \frac{1}{2} f_0 ^{t} \left( \chi^2 \sigma^2 + \epsilon^2 \left( \frac{(\lambda^2 - \lambda^4) (f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right)^2 \right) ds - f_0 ^{t} \left( - \chi \sigma \epsilon \left( \frac{(\lambda^2 - \lambda^4) (f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right) \right)} du_0 \bar{W}_t.
$$

Note that last term of the first row cancels the first term of the second row. Importantly, the second row defines a change of measure. The change of measure is

$$
\frac{d\hat{P}}{d\bar{P}} \bigg| _{t_t} \equiv \nu_t = e^{- \frac{1}{2} f_0 ^{t} \left( \chi^2 \sigma^2 + \epsilon^2 \left( \frac{(\lambda^2 - \lambda^4) (f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right)^2 \right) ds - f_0 ^{t} \left( - \chi \sigma \epsilon \left( \frac{(\lambda^2 - \lambda^4) (f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right) \right)} du_0 \bar{W}_t,
$$

where the $\hat{P}$-Brownian motion $\hat{W}$ is defined as

$$
d\hat{W}_t = d\bar{W}_t + y_t dt
$$

$$
y_t = \left( - \chi \sigma \epsilon \left( \frac{(\lambda^2 - \lambda^4) (f - f_J)}{\sigma_f \sqrt{1 - \rho^2}} \right) \right) \top.
$$

Rewriting the problem under the probability measure $\hat{P}$ yields

$$
F(Z_t) \equiv \bar{E}_t \left[ \int_{t}^{\infty} e^{f_{u} ^{t} X_u \sigma du} \right],
$$

where $X_t = -\beta + \frac{1}{2} \frac{\chi^2 + \epsilon^2}{\sigma_f \sqrt{1 - \rho^2}} (f - f_J)^2 + \chi \left( f_J - \frac{1}{2} \sigma^2 \right) + \frac{1}{2} \chi^2 \sigma^2$. For notational ease, we drop the index $j$ when defining the function $F(.)$ by keeping in mind that $\epsilon = \frac{\lambda^2}{\lambda^4}$ and $\chi = 1 - \alpha$ in our setup. We now transform this expression to obtain a $\bar{P}$-martingale. We have

$$
F(Z_t) e^{f_{u} ^{t} X_u \sigma du} + \int_{0}^{t} e^{f_{u} ^{t} X_u \sigma du} du = \bar{E}_t \left[ \int_{0}^{\infty} e^{f_{u} ^{t} X_u \sigma du} du \right] \equiv M_t,
$$

where $M$ is a $\bar{P}$-martingale. Applying Itô’s lemma to the martingale $M$ and setting its drift to zero yields the following Partial Differential Equation for the function $F(Z)$

$$
\mathcal{L}^\bar{P} F(Z) + F(Z) X(Z) + 1 = 0,
$$

where $\mathcal{L}^\bar{P}$ is the $\bar{P}$-infinitesimal generator with respect to the vector of state variables $Z$. The dynamics of the vector of state variables are defined as follows:

$$
dZ_t = \mu(Z_t) dt + \sigma(Z_t) d\hat{W}_t.
$$

Since the function $F(Z)$ is a standard transform, we rewrite PDE (91) by setting $F(Z) = e^{L(Z)}$. That is, the function $L(Z)$ solves the following PDE

$$
\nabla L(Z)^{\top} \mu(Z) + \frac{1}{2} \sigma(Z) \sigma(Z)^{\top} + X(Z) + e^{-L(Z)} = 0,
$$

where $\nabla f(Z) = \frac{\partial}{\partial Z} f(Z)$ is the gradient of the function $f(\cdot)$, $Hess(f(Z)) = \frac{\partial^2}{\partial Z^2} f(Z)$ is the Hessian matrix of the function $f(\cdot)$, and $\text{tr}(\cdot)$ is the trace operator. Because each function can be characterized by a Taylor expansion around a reference point $Z_0$, the solution to PDE (94) can be expressed in the following Polynomial
representation

\[ L(Z) = \sum_{|i|=0}^{\infty} a_i(Z_0)(Z - Z_0)^i \]

\[ = \sum_{|i| \leq m} a_i(Z_0)(Z - Z_0)^i + \sum_{|k|=m+1} a_k(Z_+)(Z - Z_0)^k \]

\[ \equiv L(Z) + R((Z - Z_0)^{m+1}), \]

where \( L : \mathbb{R}^n \to \mathbb{R} \); the multi-indices \( i, k \) are such that \( |i| = i_1 + \cdots + i_n \) and \( |k| = k_1 + \cdots + k_n \); the \( a_i(Z_0) \) coefficients characterize the partial derivatives of degree \( |i| \) of the function \( L(Z) \) at the reference point \( Z_0 \); the \( a_k(Z_+) \) coefficients characterize the partial derivatives of degree \( |k| = m + 1 \) of the function \( L(Z) \) at a point \( Z_+ \in \Lambda_{Z,Z_0} \), where \( \Lambda_{Z,Z_0} \) is the line segment connecting \( Z \) and \( Z_0 \); \( Y^1 = Y^1_1 \cdots Y^1_n \) for any \( n \)-dimensional vector \( Y \); \( L : \mathbb{R}^n \to \mathbb{R} \) is an approximation of \( L(.) \) of degree \( m \); and \( R((Z - Z_0)^{m+1}) \) is the remainder of degree \( m + 1 \). As \( m \) converges to infinity, the approximation \( \tilde{L}(X) \) converges to the true function \( L(Z) \) because the remainder \( R((Z - Z_0)^{m+1}) \) converges to zero.

The procedure to solve for the coefficients \( a_i(Z_0) \), \( |i| \in \mathbb{N} \) in (95) consists in (1) substituting (95) in PDE (94), (2) performing a Taylor expansion of the resulting expression at the reference point \( Z_0 \), and (3) setting the loadings on \( Y^0 \equiv (Z - Z_0)^0 \), \( Y^1 \equiv (Z - Z_0)^1 \), \( Y^2 \equiv (Z - Z_0)^2 \), \ldots to zero. Since this problem involves an infinite number of equations with an infinite number of unknowns, the idea is to truncate the Taylor expansion of \( L(Z) \) and that described in (2) at a degree \( m \) and solve for the coefficients \( a_i(Z_0) \), \( |i| \leq m \). That is, the latter procedure characterizes the approximation \( \tilde{L}(Z) \) defined in (97).

As noted by Benzoni, Collin-Dufresne, and Goldstein (2011), PDE (94) would admit a closed form solution of the form

\[ L(Z) = A + B^TZ \]

(98)

if the last term were absent \((e^{-L(Z)} \approx 0)\) and both the vector of state variables \( Z \) and \( X(Z) \) were affine in \( Z \). Assuming again that the last term were absent, PDE (94) would admit a closed form solution of the form

\[ L(Z) = A + B^TZ + Z^TCZ \]

(99)

if the vector of state variables \( Z \) were affine-quadratic and \( X(Z) \) were quadratic in \( Z \) (Cheng and Scaillet, 2007). It is worth noting that, even in a single-agent economy where the fundamental \( \tilde{f} \) and the mean-reversion speed \( \tilde{\lambda} \) have constant diffusions

\[ d\tilde{f}_i = \tilde{\lambda}_i (\tilde{f} - \tilde{f}_i) dt + \sigma_f d\tilde{W}_i \]

(100)

\[ d\tilde{\lambda}_i = \kappa (\tilde{\lambda} - \tilde{\lambda}_i) dt + \sigma_{\lambda} d\tilde{W}_i, \]

(101)

the vector of state variables belongs to a class of processes that is more complex than the affine-quadratic class. Indeed, augmenting the vector of state variables with \( \tilde{f}^2, \tilde{\lambda}^2 \), and \( \tilde{f}\tilde{\lambda} \) and computing the drift and variance-covariance matrix of the augmented vector involves terms of order larger than two. This shows that, even in a single-agent economy, time-varying mean-reversion implies a solution to PDE (94) that is more complex than the quadratic form in (99).

To solve our problem, we do however choose to truncate the Taylor expansion of \( L(Z) \) around the reference point \( Z_0 = (\tilde{f}, \tilde{\lambda}, \gamma_{ss})^T \) at a degree \( m = 2 \), where the steady-state uncertainty \( \gamma_{ss} \) solves \( \frac{\partial \gamma_{ss}}{\partial ss} |_{\tilde{f}=\bar{f}} = 0 \) and hence satisfies \( \gamma_{ss} = \frac{(1-\sigma^2)e^2}{2\kappa} \). That is, we consider a quadratic approximation of the form

\[ \tilde{L}(Z) = A + B^TZ + Z^TCZ, \]

(102)

where \( Z^T = (\tilde{f}^A, \tilde{\lambda}^B, \gamma)^T \) because adding terms of higher order does not significantly affect (1) the
The residuals of the PDE\textsuperscript{24} and (2) the value of the approximation. Figure 6 depicts the residuals of the PDE when $\epsilon = 0$ and $\chi = 1 - \alpha = -2$ for different values of the state variables. Note that we choose to illustrate the PDE residuals when $\epsilon = 0$, as opposed to $\epsilon \in \{1/3, 2/3, 1\}$, because this parametrization yields the largest residuals. This figure shows that our approximation is accurate for a large range of values of the state variables.

A.4 Calibration to the U.S. Economy

In the model, each agent models the dynamics of the dividend growth and the expected dividend growth using three sources of risk: a dividend growth shock, an expected dividend growth shock, and an information shock. Because agents consider three shocks but we only observe two time-series, the model cannot be estimated by Maximum-Likelihood. Moreover, the fact that the fundamental features a stochastic mean-reversion speed implies that only few moments can be computed in closed form and thus we cannot apply the Generalized Method of Moments either.

We choose therefore to estimate the parameters of the model by the Simulated Method of Moments (SMM). The data that we used spans Q1:1969-Q3:2014 and was obtained from the Federal Reserve Bank of Philadelphia. First, we simulate 20,000 paths of the state variables $\zeta$, $\tilde{f}$, $\tilde{\lambda}^B$, and $\gamma$ over a 45-year horizon at weekly frequency. The horizon is chosen to match its empirical counterpart, and the relatively high frequency is chosen to mitigate simulation errors generated by the Euler discretization scheme. Second, we record the log-dividend growth rate and the expected dividend growth rate at a quarterly frequency for each simulation. Third, we compute the following moments and their average across simulations:

- the mean of the log-dividend growth rate,
- the volatility of the log-dividend growth rate,
- the 1-quarter autocorrelation of the log-dividend growth rate,
- the mean of the expected dividend growth rate,
- the volatility of the expected dividend growth rate,
- the 1-quarter autocorrelation of the expected dividend growth rate,

\textsuperscript{24}The residuals of the PDE are simply obtained by plugging the approximation (102) in PDE (94). By definition, a closed form solution to PDE (94) yields residuals that are equal to zero, whereas a good approximation yields residuals that are close to zero.
• the correlation between the log-dividend growth rate and the expected dividend growth rate,
• the 2-, 3-, and 4-quarter-ahead variance ratios of the log-dividend growth rate,
• the 2-, 3-, and 4-quarter-ahead variance ratios of the expected dividend growth rate for each simulation.

The vector of parameters, \( \Theta = (\sigma, \tilde{f}, \sigma_f, \rho, \kappa, \bar{\lambda}, \Phi, \phi)^\top \), is chosen to minimize the following criterion

\[
\Theta = \arg\min_{\Theta} [M^e - M(\Theta)]^\top W [M^e - M(\Theta)],
\]

where \( M(\Theta) \) is the 13-dimensional vector of simulated moments, \( M^e \) is its empirical counterpart, \( W = \Omega^{-1} \) is the optimal weighting matrix, and \( \Omega \) is the variance-covariance matrix (computed across simulations) of the vector of moments \( M(\Theta) \).
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