innovative dynamic pricing: the potential benefits of early-purchase reward programs

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Abstract

We propose a scientific, game-theoretic model in which a monopolist sells a fashion product in a market consisting of strategic consumers. Research papers on this subject have studied the adverse consequences of strategic consumer behavior, and have proposed a variety of mechanisms to counteract this phenomenon. In this work, we introduce a very broad class of mechanisms which we refer to as early-purchase reward (EPR) programs. Such class includes, but is not limited to strategies such as price-matching, price commitment, and capacity rationing. Confining ourselves to a simple but tractable model, we are able to obtain a complete analytical characterization of the optimal EPR program that a seller should offer to its consumers. Such a program maintains a structure that consists of two components: (1) a promised refund ("participation bonus") that segments the market over the course of the season, and (2) a "modified price matching guarantee" that serves as a lever to counteract strategic consumer behavior. The latter component protects consumers against price drops, but charges them for the fact that they get to enjoy the product by consuming it earlier rather than later in the season. We find that optimal EPR programs are particularly valuable when the inventory and the degree of fashion are high. We have also conducted a numerical analysis to further compare an EPR program (heuristic) to an optimal price-matching strategy, in a rich modelling framework that consists of market size uncertainty, production costs, mixture of strategic and non-strategic consumers, and more. Our results suggest that EPR programs can be advantageous in settings involving modest-to-high degrees of fashion and high degrees of market size uncertainty, regardless of the percentage of strategic consumers in the market.

Key words: dynamic pricing; strategic consumer behavior; revenue management; game theory.

1 Introduction

The phenomenon of strategic consumer behavior in the context of pricing theory has been a topic of discussion since the seminal paper of Coase (1972). Using qualitative arguments, and focusing on a monopolist selling a durable good, Coase argued that the monopolist cannot effectively exercise price discrimination, because if the consumers rationally anticipate a price drop at any point of time in the future, they will postpone their purchase to that time - the essence of *strategic behavior*. Following this logic, Coase conjectured that the only price at which the monopolist can expect to sell its product is the *clearance price* for the quantity it has on hand. This means that strategic consumer behavior results not only in the lack of ability to exercise discrimination, but the monopolist completely loses market power in the sense that it cannot even charge the best *fixed* price. One of the remedies to the loss of market power that Coase proposed, relevant to our current research, is an inter-temporal *price-matching* mechanism: the monopolist lists the optimal fixed price along with a buy back guarantee to its consumers. Effectively, under such an arrangement, the monopolist agrees to match the price at any time in the future if a price decrease is made. The consequence of such mechanism is noteworthy – not only that the consumers are no longer concerned about the timing of their purchase, but it will also not be optimal for the monopolist to reduce the price below the optimal fixed-price level, since it would have to issue a refund to the consumers who purchased earlier. Thus, the monopolist regains market power.

Stokey (1979) considered a monopolist selling a product in a market consisting of strategic consumers. In her model, the monopolist is able to credibly commit to a price path during the sales horizon. Under this pricing approach, which we will refer to as a *price-commitment* strategy, Stokey showed (for the case of zero production cost) that it is optimal for the monopolist to offer the best possible fixed price. In other words, strategic consumer behavior continues to eliminate the monopolist's ability to price discriminate, but the monopolist regains its market power via price commitment. Landsberger and Meilijson. (1985) studied a model similar to that of Stokey, but added a model feature that is fundamentally important to our current research. Specifically, they considered a situation in which the consumers experience loss in their utility when they purchase the product later in the course of the sales period. This is typically the situation when a product is *fashion-like*, and the consumers strongly prefer buying it earlier than later. Under such condition, the authors show that the seller ends up selling the product at different price levels; in other words, the seller exercises some level of price discrimination. When price commitment is used as a mechanism to fully control the consumers' expectations regarding future price drops, the consumers can be discouraged from waiting. However, since price discounts must be sufficiently low in order to achieve such impact, the seller's ability to effectively discriminate may vanish.

The research on the topic of strategic consumer behavior has seen significant growth over the past decade. Generally, the fresh perspective found in the majority of the research papers published in our literature, has to do with the exploration of dynamic pricing settings of *seasonal* (or fashion-like) goods, where a seller attempts to maximize revenue (or profit) by selling a *limited capacity*. In order to avoid an unnecessarily long discussion of the literature, we offer below a limited but focused review. We refer the interested readers to surveys such as Aviv and Vulcano (2012), as well as several other chapters in Netessine and Tang (2009). Additional literature reviews can further be found in a few of the more recent papers we cite below.

Since the central focus of the current research is on identifying a mechanism for counteracting the adverse impact of strategic consumer behavior, it is of useful to consider the existing literature through this lens. One broad class of mechanisms that has been studied in the literature can be described as *capacity control*. For example, Su (2007) and Liu and van Ryzin (2008) analyze the effectiveness of capacity rationing strategies, in which the seller limits the number of units sold to a level lower than it would set if consumers were non-strategic. The power of this approach is in creating shortage risk for the consumers. Specifically, a consumer that contemplates buying at premium price vs. waiting for a discount may feel inclined to purchase early due to the concern that the product might not be available anymore later in the season. Another class of mechanisms falls into the broad category of *supply chain coordination*. For instance, Su and Zhang (2008) studied the influence that certain supply chain contracts have on the perceptions of strategic consumers regarding future price changes and product availability. Interestingly, they demonstrated that decentralized but appropriately coordinated supply chains may be able to perform better than centralized supply chains, when faced with strategic consumers. Cachon and Swinney (2009, 2011) consider the value of quick response and product design strategies in a supply chain, in face or in the absence of strategic consumers. For example, one of the key findings of Cachon and Swinney (2009) is that the seller's ability to replenish its inventory just before the season and after observing some market information, can mitigate strategic behavior since it reduces the chance of clearance sales. The underlying logic behind this is that if the seller does not anticipate a high demand, then it will not order additional inventory, leading to either low product availability or small markdowns at the end of the season. On the other hand, if sales at the premium price are expected to be high, then the seller may bring additional units to the store, but continue to offer the product at a relatively high price. Consequently, in equilibrium, consumers do not expect to get the product at a substantial discount at the end of the season.

A third class of mechanisms can be broadly described as *information-based control*. For instance, Yin et al. (2009) compare two product display settings: one in which the seller places all units on the shelf and thus inventory is fully visible to the consumers, and one in which the seller displays only a unit at a time. The authors find that the latter approach results in an equilibrium behavior that leads to better revenue performance. They argue that this outcome results from an increased shortage risk perception among consumers, which suppresses strategic behavior. Jerath et al. (2010) explores the impact of opaque selling strategies, in which a seller hides certain service attributes of its products when offering them later in the season. Such practice is used by airlines (e.g., where the departure times for flights are not fully disclosed at the time of booking) and in the hotel industry (e.g., disclosing a hotel's quality level without revealing its specific identity). Due to their nature, opaque strategies can potentially mitigate strategic waiting, but similarly they may negatively affect demand at the later part of the sales season. Another related model has been proposed by Aviv et al. (2013), who study a setting in which a seller offers a fashion good, and has an opportunity to obtain significant demand information via early sales observation. In one part of their study, the authors demonstrated that when all consumers are strategic, it may be better for the seller to proactively avoid learning. In other words, it may not be beneficial for the seller to integrate up-to-date sales information into the pricing process. They explain this result by pointing to a phenomenon called "information shading" – where under conditions of market equilibrium, more strategic consumers end up postponing their purchases, as such action deprives the seller from obtaining valuable demand information. In turn, this increases the likelihood of higher price markdowns at the end of the season. When a seller commits to not learn, information shading is eliminated, and consumers expect lower discounts. Continuing with the class of information related models, we refer the interested readers to several recent papers on strategic consumer behavior in the presence of price information (Cachon and Feldman 2013), product quality considerations (Yu et al. 2013a, Swinney 2011), advance selling (Prasad et al. 2011), and social learning (Yu et al. 2013b, Papanastasiou and Savva 2014).

Turning our focus a step closer to the agenda of our current research, we note a fourth class of mitigating mechanisms, which we refer to as *pricing control*. Recent research papers have explored the value of price-commitment strategies in the presence of strategic consumer behavior. For example, Aviv and Pazgal (2008) demonstrated the positive benefit that price commitment can bring in the sales of seasonal goods, using a model in which consumers arrive to the stores at random times, and the seller has limited and non-replenishable capacity. The authors argue that the benefits of price commitment can be significant even in situations where the inventory level is low, and strategic waiting is theoretically minimal due to the presence of shortage risk. They explain this phenomenon via the examination of optimal pricing behavior: when a seller has limited inventory, the seller generally prefers to deploy a "betting" strategy; i.e., price the product high with the hope of capturing customers with high valuations for the product; then, plan to reduce prices later in the season. However, following this type of strategy is obviously impractical when the consumers are strategic. Cachon and Swinney (2009), mentioned above, caution against adopting price-commitment strategies without giving full consideration to the supply chain's agility. Specifically, they argue that when a seller can deploy quick response, commitment is not particularly useful in counteracting strategic consumers, and in fact can lead to a dramatic loss in revenue performance. Aviv et al. (2013), mentioned above, also study the potential benefits of price-commitment strategies in the sales of fashion goods. Their study demonstrates that the ability to reduce strategic consumer behavior through commitment can significantly outweigh the loss of flexibility to react to early sale information. Lai et al. (2010) study the potential benefits of inter-temporal (posterior) price-matching strategies. As their paper played a valuable role in our current research, we discuss it in significant details in §5. Like price-commitment strategies, price-matching guarantees can be powerful in suppressing strategic consumer behavior. But, similarly, they also impose limitations on the ability of a seller to effectively price discriminate.

The purpose of this research paper is *theoretical* in nature. When looking at the price control mechanisms mentioned above, we note that they appear to be extreme in nature: either a *rigid* commitment to a price path, or an agreement to pay a *full* refund in case of a price drop. But when products are fashion-like, in the sense that they experience a drop in consumer valuations over the course of the sales season, it might be counter-productive for a seller to adopt these types of pricing policies. For instance, in order to discourage a strategic consumer from waiting to buy a high-end swim suit at the end of the summer season, the seller may not need to offer a full price match early on. This is due to the fact that the decline in the consumer's utility when getting the product later, rather than earlier in the season, already serves as an incentive to purchase early at the premium (main season) price. Thus, we propose a broader class of pricing-control mechanisms, to which we shall refer as *early-purchase reward (EPR) programs*. Under EPR, the seller publishes, along with the listed premium price, an agreement to issue a credit (reward) to any consumer who purchases the product at that price. The reward may be dependent on the time of purchase, the sales realizations during the season, and the price of the product at the end of the season. Obviously, price-commitment and price-matching strategies are special cases of the broader EPR class (see specific details in §4): A price commitment can be described as an EPR program that guarantees to issue a very large payment to consumers if the price markdown at the end of a season will be different than a particular value. This effectively means that the seller becomes fully committed to that published price discount. Alternatively, an EPR program that offers a reward that equals to the future markdown is identical to a price-matching strategy. The EPR class also includes the null case – offering no rewards – which brings us to the baseline case in which the seller applies no pricing-control mitigating mechanism. In fact, the EPR class includes certain capacity-control mechanisms too. Consider a situation in which a seller would like to create rationing risk by limiting the number of units sold in the season (say, x units). This can be administered via an EPR program that guarantees a prohibitively high credit to consumers in case that the total sales exceed x.

We essentially focus on a single, yet thought-provoking scientific puzzle: the identification of the best EPR program. In this vein, our approach is prescriptive, rather than normative, and our interest is primarily in delivering managerial insights into the choice of innovative pricing policies in markets of fashion-like products, as described above. Indeed, our key challenge in the early phase of this research project was to identify a simple, yet technically-tractable, model that can lead to clear and insightful results. Following this perspective, the underlying objective of this work is to stimulate an academic and practical discussion on the potential value of innovative pricing-control policies in markets where strategic consumers are present.

The rest of the paper is organized as follows. In Section 2, we introduce our main model and identify the optimal early-purchase reward program. Section 3 gauges the benefits of adopting the optimal reward program in comparison to the baseline setting in which the seller selects the prices at the beginning and the end of the season, without using any mitigating strategy. In Section 4 we further compare the performance of an optimal early-purchase reward program to the performance of an optimal price commitment strategy and the optimal inter-temporal price-matching strategy. In Section 5, we illustrate the potential advantage of a heuristic reward program over a posterior price-matching strategy, using the broad framework of Lai et al. (2010). To conclude, we provide a brief summary of our key findings in Section 6. All proofs, with the exception of those omitted due their simplicity, are given in the Appendix.

2 Early-Purchase Reward (EPR) Programs: Model and Analysis

We develop a game-theoretic model, under a continuous time setting, to study the research questions raised in the introduction. Consider a seller who continuously sells a fixed inventory Q through a relatively short selling season, which spans from time 0 to time 1. At the beginning of the selling season, the seller sets a price p_1 ("premium price"), and maintains this price until the end of the selling season, time 1, at which point the seller has an opportunity to change the price to p_2. Respectively, we refer to the time when the market faces price p_1 as the main season, and the time when the price p_2 is offered as the "end-of-season".

We now turn to the central idea of this paper. When setting the premium price, the seller also publishes an *early-purchase reward program*, promising a non-negative back payment of $r(t, p_1, p_2, I) \in [0, p_1]$ to any consumer who purchases the product at the premium price p_1, at time t. To maintain as much generality as possible, we allow the reward program to depend on all sales-path information: the prices (p_1, p_2), the time of the individual consumer's purchase (t), and the complete sales path which is reflected by the inventory level $I \doteq \{i(t) : t \in [0, 1)\}$). Furthermore, for all practical purposes, we restrict our attention to reward programs that are non-increasing in p_2. This means that larger price markdowns at the end of the season should lead to larger (or equal) paybacks to the consumers. Below, we shall use the letter R as a shorthand notation for the program; i.e.,

$$R \doteq \{r(t, p_1, p_2, I) : t \in [0, 1]\} \text{ (Reward Program)}$$

The fact that the reward program is contingent on p_1, p_2, and I implies that the reward program could be implicitly contingent on other measures inter-dependent with those three parameters (e.g. the probability of actually obtaining the product at the end of the season).

Consumers are infinitesimally small and arrive continuously at a constant rate of λ during the main season. The consumers are heterogeneous in their valuations, which decline over the season. To this end, we consider a market of consumers with *base valuations* (i.e., valuation if they purchased the product at time $t = 0$) distributed independently and uniformly over $[0, 1]$. Furthermore, a consumer with base valuation v

receives a value of size

$$V(v,t) = ve^{-\alpha t},$$

if he makes a purchase at time t. The exponential decline rate, $\alpha \geq 0$, is assumed to be fixed across the population. The parameter α plays a pivotal role in this study, and we shall informally use it in the proxy measure

$$\delta \doteq 1 - e^{-\alpha} \in [0,1) \text{ (“Degree of fashion”)},$$

reflecting how *fashion-like* the market for the product is. Note that δ represents the decline in valuation from the very beginning of the season ($t = 0$) to its end ($t = 1$). Thus, when δ is significant (high degree of fashion), a consumer would gain a significantly higher value from consuming the product earlier than later. Conversely, when δ is low (low degree of fashion), a consumer will receive similar value from the product, regardless of the time of purchase.

For technical convenience, we assume that upon arrival, a strategic consumer chooses either to purchase the product immediately at the premium price p_1, or to wait for the end of the season. However, the consumer will not consider other purchase times in between his arrival time and the the end of the season. This behavioral assumption should not be of concern, as we shall see that it is indeed optimal for the consumers to follow when the seller offers the optimal reward program; see Proposition 7 in §2.4.

We model the interaction between the seller and the consumers as a sequential game. As mentioned above, the seller first posts the premium price (p_1) and the reward program details (R) at the very beginning of the season ($t = 0$). Then, the consumers and the seller are further engaged in a Stackelberg-form game (the *“subgame”*), as follows. The consumers – the Stackelberg leaders – arrive at the store, and contemplate between purchasing immediately and waiting for the end of the season. Since the consumers' actions influence the end-of-season prices and the rewards paid later, their decisions become complexly inter-dependent. In fact, the study of the game between the consumers will be the starting point of our analysis; see §2.1. At the end of the season, after observing consumers' purchasing behavior and the sample path of the inventory, the seller – the Stackelberg follower – can adjust the price from p_1 to p_2; see §2.2. At the end of the season, all consumers who did not purchase the product will attempt to do so if their valuation at that time is higher than the price p_2. If the number of consumers who request the product exceeds the available inventory units, then a random allocation is made on the basis of equal probabilities. After studying the subgame, we will turn our attention to the identification of the optimal reward program in §2.3, which will follow by the analysis of the seller's best premium-price decision and the calculation of its optimal revenue performance; see §2.4. As often done in the analysis of game theoretic models of this type, we assume that all of the model parameters are common knowledge. Furthermore, we only consider the situation in which the seller and consumers are risk neutral, aiming to maximizing their expected payoffs (revenue and surplus, respectively).

2.1 The Consumers' Optimal Purchasing Decisions

Consider any premium price p_1 and an announced reward program R. To predict the consumers' purchasing decisions in equilibrium, we assume that a typical consumer compares the surplus gained by an immediate purchase with the expected surplus gained if he decides to wait for a possible discount. First, Proposition 1 below allows us to characterize a consumer's best action in response to *any* anticipated end-of-season pricing scheme $p_2(I)$ and *any* arbitrary purchasing behavior adopted by other consumers. We demonstrate that when the store is "open" (i.e., when the inventory is still available) at time t, the consumer will adopt a threshold-form policy with a unique threshold value.

Proposition 1 *Consider a consumer arriving at time t with a base valuation v, and suppose that the store is open. Then, for any anticipated end-of-season pricing policy $p_2(I)$ and any purchasing behavior of all other consumers in the market, it is optimal for the consumer to follow a threshold policy with a unique threshold value $\theta(t) \in [0,1]$. Namely, the consumer should purchase a unit at the price p_1 if his base valuation is higher than that threshold (i.e. $v > \theta(t)$); otherwise, the consumer should wait for the end of the season.*

The proposition above excludes the possibility of mixed-strategy equilibria in the consumers' game. Thus, an equilibrium – if exists – can be described by the pure strategy functional form $\Theta \doteq \{\theta(t) : t \in [0,1]\}$. In fact, the existence of an equilibrium is established in the following proposition.

Proposition 2 *The consumers' game possesses a Nash equilibrium Θ.*

Note that while an equilibrium in the consumers' game exists, it is not necessarily unique. In fact, we have actually observed multiple equilibria for certain *arbitrary* forms of the reward program. Yet, it should not be a concern, as we shall demonstrate that under the *optimal* reward program and the optimal premium price, the equilibrium for the consumers' game is actually unique (see §2.4).

2.2 The Seller's End-of-Season Pricing Decision

Next, we describe the seller's end-of-season pricing problem. At time $t = 1$, if inventory is left, the seller needs to weigh the pros and cons of a price change. On one hand, the seller may want to entice purchases via a price markdown, as it is practically the last chance to generate additional revenue. However, a price reduction may force the seller to issue paybacks to the consumers who purchased at the premium price, as promised in the published early-purchase reward program. Our analysis below quantifies these opposing factors.

For a given purchasing policy Θ followed by the consumers, we define the available inventory at time t by $i(t,\Theta) \doteq \left(Q - \int_0^t \lambda(1-\theta(t))\,dt\right)^+$, and the inventory trajectory over the first period by $I(\Theta) \doteq \{i(t,\Theta) : t \in [0,1]\}$. In particular, let $q(\Theta) \doteq i(1,\Theta)$ be the remaining inventory at the end of the season, just before a price change is made. Thus, for any given end-of-season price p_2, the expected number of additional units sold would be

$$x_2(\Theta, p_2) \doteq \min\left(q(\Theta), \int_0^1 \lambda(\theta(t) - p_2 e^{\alpha})^+\,dt\right),$$

where the second term in the minimization operator represents the demand induced by p_2. The seller's end-of-season pricing problem (relevant in the case $q(\Theta) > 0$) can be written in the following form:

$$\pi_2(\Theta, R) \doteq \max_p \left\{ p x_2(\Theta, p) - \int_0^1 r(t, p_1, p, I(\Theta))\,\lambda(1-\theta(t))\,dt \right\}. \tag{1}$$

In fact, the following proposition demonstrates that under optimal selection of the price p_2, $x_2(\Theta, p_2) = \int_0^1 \lambda(\theta(t) - p_2 e^{\alpha})^+\,dt$.

Proposition 3 *The price p_2 will never induce an end-of-season demand that is larger than the remaining level of inventory $q(\Theta)$.*

The proof of the proposition is trivial, and hence omitted. Clearly, if a price p_2 induces a demand that is higher than $q(\Theta)$, then the seller could increase the price slightly to generate both higher revenues at the end of the season and lower (or equal) the reward paybacks issued to those who purchased at the premium-price. One of the implications of the last proposition is that, in their actions, consumers face price consequences rather than shortage consequences.

The selection of an optimal end-of-season price $(p_2(\Theta, R))$ is challenging for two reasons. First, the seller's end-of-season revenue depends on the reward program R, which can be complex in p_2. Second, the revenue function is contingent on the consumers' behavior Θ, which in and of itself can be an irregular function. Consequently, we shall complete the discussion on how to determine of the optimal value of p_2 after characterizing the functional forms of R and Θ.

2.3 The Optimality of Surplus-Matching Reward (SMR) Programs

We now turn our focus to the beginning of the season, when the seller needs to publish the price p_1 and the reward program R. Recall that at this stage, the seller acts first, and immediately thereafter a Stackelberg subgame is initiated, where the consumers adopt a pure purchasing strategy Θ, and the seller responds with a price $p_2(\Theta, R)$ thereafter, by solving (1). Using the notation and results presented earlier, we introduce the following consumer surplus functions. First, if a consumer arriving at time t with a base valuation v decides to purchase immediately, then his expected surplus would be

$$s_1(v, t|\Theta, R) \doteq ve^{-\alpha t} - p_1 + r(t, p_1, p_2(\Theta, R), I(\Theta)). \tag{2}$$

In contrast, if this consumer decides to wait for the end of the season, then his expected surplus would be

$$s_2(v|\Theta, R) \doteq \beta(\Theta, p_2(\Theta, R)) \cdot \left(ve^{-\alpha} - p_2(\Theta, R)\right)^+, \tag{3}$$

where the function $\beta(\Theta, p_2)$ represents the probability for this consumer to actually obtain the product. Specifically, $\beta(\Theta, p_2) \doteq \min\left\{\frac{q(\Theta)}{\int_0^1 \lambda(\theta(t) - p_2 e^{\alpha})^+ dt}, 1\right\}$; this probability is based on a random-allocation mechanism that gives each interested consumer an equal chance for obtaining a unit. We are now ready to establish one of our key findings. Note that in view of Proposition 3 the value of β is 1. However, for the purpose of generality (which we will need in our comparative studies in §4), we present (3) in its probabilistic form.

Theorem 1 *Without loss of optimality, the seller can restrict its attention to early-purchase reward programs that satisfy the following equality:*

$$s_1(\theta(t), t|\Theta, R) = s_2(\theta(t)|\Theta, R) \text{ for all } t \in [0, 1]. \tag{4}$$

The theorem shows that the seller can focus its attention on reward programs that lead to an equilibrium in which the consumers' threshold maintains an *exact* balance between the two surpluses s_1 and s_2 across the entire sales horizon. We correspondingly name this class of programs *surplus-matching reward (SMR) programs.* It is noteworthy that this theorem is not trivial. For example, the special cases of $R = 0$ (no reward offered) and $R = (p_1 - p_2)^+$ (price matching) do not necessarily satisfy the condition (4). In additional to providing a qualitative framework for explaining the nature of optimal reward programs, Theorem 1 greatly simplifies the challenge in finding such programs.

2.4 The Optimal Premium Price and Reward Program

In this section we discuss the seller's optimal choice of the premium price and the early-purchase reward program, restricting our attention to SMR programs in view of Theorem 1. We first establish the following property.

Proposition 4 *Without loss of optimality, the seller can focus his attention on SMR programs and premium prices for which the induced demand in the main season (i.e.* $\int_0^1 \lambda(1-\theta(t))\,dt$*) is smaller than or equal to the initial inventory level* Q*.*

This result is intuitively clear. If the seller offers a price p_1 and a reward program that induce more demand than its initial inventory, then it could alternatively offer a higher price at a level that would make the demand in the first period equal to Q. The revenue would then increase, while still preventing strategic waiting (as no inventory is expected to be left for the end of the season).

Let's denote $\Theta(R, p_1)$ and $p_2(R, p_1)$ as the subgame equilibrium pair given an arbitrary premium price p_1 and an SMR program R. Then, utilizing Proposition 4, we can present the seller's optimization problem as follows:

$$\Pi(Q) \doteq \max_{p_1, R} \left\{ p_1 \int_0^1 \lambda(1-\theta(t, R, p_1))\,dt + \pi_2(\Theta(R, p_1), R) \right\}. \tag{5}$$

Despite the considerable simplification brought by Theorem 1, the seller's initial-stage decision problem (5) is still complicated by two factors. First, the determination of the optimal reward program requires a search over the space of possible reward *functions*. Second, we must take into account the intricate interdependence between p_1, R, and Θ^E. The way we propose to approach this challenge is by treating the seller's problem as a *segmentation problem*. Specifically, instead of searching directly for the best values of p_1 and R as presented in (5), we conduct the search over the space of surplus-matching reward programs (i.e., the functions Θ). As a standard procedure, we first conjecture the structure of the optimal solution. Specifically, we propose that in equilibrium, the consumers' purchasing behavior Θ is non-decreasing with respect to time (i.e., $\partial\theta(t)/\partial t \geq 0$) and continuously differentiable in $[0, 1)$. In other words, as time gets closer to the point of discount, a consumer with a given base valuation will be less inclined to purchase immediately. Such property has been established across a variety of papers in the literature; see, e.g., Aviv and Pazgal (2008).

Our segmentation approach works as follows. For any given candidate function Θ we look for the best value of p_2 that maximizes the seller's revenue performance. When we do so, we deploy the result of Theorem 1, which allows us to focus on surplus-matching reward programs. Specifically, we argue that the total revenue collected by the seller is given by

$$\lambda \int_0^1 \left(\theta(t) e^{-\alpha t} - \left(\theta(t) e^{-\alpha} - p_2\right)^+\right) \cdot (1 - \theta(t))\, dt + \lambda \int_0^1 p_2 \cdot \left(\theta(t) - p_2 e^{\alpha}\right)^+ dt \tag{6}$$

where p_2 and Θ must satisfy the condition

$$\lambda \int_0^1 (1 - \theta(t))\, dt + \lambda \int_0^1 \left(\theta(t) - p_2 e^{\alpha}\right)^+ dt \le Q. \tag{7}$$

in view of Propositions 4 and 3. We now present an intermediate result.

Proposition 5 *Let Θ be a non-decreasing and continuously differentiable function in t. Furthermore, suppose that $\lambda \int_0^1 (1 - \theta(t))\, dt \le Q$, and consider the problem of maximizing (6) over p_2, subject to the constraint (7). The solution to this problem is given by*

$$p_2^*(\Theta) = \begin{cases} \min\left\{\max\left\{\frac{1}{2}e^{-\alpha}, \left(1 - \frac{Q}{\lambda}\right)^+ e^{-\alpha}\right\}, \theta(1) e^{-\alpha}\right\} & \text{if } \theta(0) > 1 - \frac{Q}{\lambda} \\ \min\left\{\max\left\{\frac{1}{2}e^{-\alpha}, \tilde{p}_2(\Theta)\right\}, \theta(1) e^{-\alpha}\right\} & \text{if } \theta(0) \le 1 - \frac{Q}{\lambda} \end{cases} \tag{8}$$

where $\tilde{p}_2(\Theta)$ is defined as the unique value of p_2 that satisfies constraint (7) with an equality.

At this stage, it is instructive to note that the proposition does not guarantee that a solution of the type (8) would actually exist in equilibrium, as Θ must be the consumers' best response to the price path and reward functions – all yet to be determined. But, setting this issue aside for a moment, let us proceed to the following result.

Proposition 6 *Consider the problem of maximizing (6) over p_2 and Θ, subject to the constraints (7) and Θ being a non-decreasing and continuous function in t. The solution to this problem is given by: (i) if $Q \ge \frac{\lambda}{2}$, then $\Theta^* \equiv \frac{1}{2}$ and $p_2^* = \frac{1}{2}e^{-\alpha}$; effectively, the seller will sell its products only in the main season. (ii) if $\frac{\lambda}{2}\left(\frac{e^{-\alpha}-1+\alpha}{\alpha}\right) \le Q < \frac{\lambda}{2}$, then $\Theta^* = \frac{1}{2} + \frac{\alpha}{e^{\alpha}-1} e^{\alpha t}\left(\frac{1}{2} - \frac{Q}{\lambda}\right)$ and $p_2^* = \theta^*(1) e^{-\alpha}$. (iii) if $Q < \frac{\lambda}{2}\left(\frac{e^{-\alpha}-1+\alpha}{\alpha}\right)$, then $\Theta^* = \min\left\{\frac{1}{2} + \frac{\alpha}{e^{\alpha\rho}-1} e^{\alpha t}\left(\frac{\rho}{2} - \frac{Q}{\lambda}\right), 1\right\}$, where ρ is the the unique solution to the equation $\frac{\rho}{2} - \frac{1}{2\alpha}\left(1 - e^{-\alpha\rho}\right) = \frac{Q}{\lambda}$. Here, $p_2^* = \theta^*(\rho) e^{-\alpha} = \theta^*(1) e^{-\alpha} = e^{-\alpha}$.*

Proposition 6 plays a critical role in our analysis. Essentially, rather than establishing an equilibrium, it identifies an *upper bound* on the seller's revenue performance. Again, it is crucial for the reader to appreciate that this upper bound on performance is not necessarily attainable. However, if one identifies a pair (p_1, R) that leads to such performance, then this pair is optimal, indeed. This brings us to our key result:

Theorem 2 *Let us define the "planned segmentation" function*

$$y(t) \doteq \begin{cases} \frac{1}{2} & Q \geq \frac{\lambda}{2} \\ \min\left\{\frac{1}{2} + \frac{\alpha}{e^{\alpha\rho}-1} e^{\alpha t}\left(\frac{\rho}{2} - \frac{Q}{\lambda}\right), 1\right\} & Q < \frac{\lambda}{2} \end{cases}$$

where ρ is defined as in Proposition 6. Then, it is optimal for the seller to select its first-period price p_1^* at any level in the range $[y(0), 1]$ and offer the following surplus-matching reward program:

$$r^*(t, p_1, p_2) = p_1 - y(t) \cdot e^{-\alpha t} + \left(y(t) \cdot e^{-\alpha} - p_2\right)^+ \geq 0. \tag{9}$$

Consequently, in the subgame, consumers arriving at time t will adopt the unique threshold function $\theta^*(t) = y(t)$, and the seller will sequentially set its second period price $p_2^* = y(1) \cdot e^{-\alpha}$. The seller's optimal total revenue is given by

$$\Pi^*(Q) = \begin{cases} \frac{\lambda}{4} \cdot \frac{1-e^{-\alpha}}{\alpha} & Q \geq \frac{\lambda}{2} \\ \frac{\lambda}{4} \cdot \frac{1-e^{-\alpha}}{\alpha} - \frac{\lambda}{4} \cdot \frac{\alpha}{e^{\alpha}-1}\left(1 - \frac{2Q}{\lambda}\right)^2 & \frac{\lambda}{2}\left(\frac{e^{-\alpha}-1+\alpha}{\alpha}\right) \leq Q < \frac{\lambda}{2} \\ \frac{\lambda}{4} \cdot \alpha \cdot \left(\rho - \frac{2Q}{\lambda}\right)^2 & Q < \frac{\lambda}{2}\left(\frac{e^{-\alpha}-1+\alpha}{\alpha}\right) \end{cases}$$

Theorem 2 demonstrates that the optimal surplus-matching reward program is designed in the following way. The first, *non-contingent* component $p_1 - y(t) \cdot e^{-\alpha t}$, effectively offers a time-dependent discount that brings the net price to the level $y(t) \cdot e^{-\alpha t}$. This results in a price path that makes the product *affordable* for all customers within the optimal planned segment $\{[y(t), 1] : t \in [0, 1]\}$. The second, *contingent* component $(y(t) \cdot e^{-\alpha} - p_2)^+$, offers to match the end-of-season price, but up to the level $y(t) \cdot e^{-\alpha}$ *only*. Note that if such match payment is issued to a consumer who purchased a unit at time t, this consumer's net price paid would be $p_2 + y(t) e^{-\alpha t} - y(t) e^{-\alpha}$. Interestingly, this suggests that the seller utilizes a logically-fair mechanism that – on one hand – protects the consumer against price drops, but – on the other hand – charges the consumer for the fact that he gains a larger value by consuming the product earlier (at time t) rather than later (at the end of the season). Since the personal valuation of a consumer is private knowledge, the latter charge is given by $y(t) e^{-\alpha t} - y(t) e^{-\alpha}$, which represents the "threshold consumer"; i.e., one with base valuation $y(t)$. This key feature of the optimal early-purchase reward program leads us to conjecture that its value compared to programs such as inter-temporal price matching guarantees, should be higher as we deal with products that have higher degrees of fashion; see our studies in §4.

As the theorem shows, the optimal reward program (9) induces a purchasing behavior $y(t)$, and it is straightforward to see that $y(t)$ is non-decreasing in t; in particular, $y(t)\, e^{-\alpha} \leq p_2^* = y(1)\, e^{-\alpha}$. This means that the consumers do not expect the contingent portion of the reward payment to be paid, but only the non-contingent payback amount $p_1 - y(t) \cdot e^{-\alpha t}$. In other words, the contingent component of the program serves as an incentive mechanism for the seller to not drop the price below $y(1)\, e^{-\alpha}$ at the end of the season, but it is not actually exercised in equilibrium. This leads to the following observation.

Proposition 7 *The optimal early-purchase reward program (9) eliminates strategic waiting. In fact, in the equilibrium identified in Theorem 2, if a particular consumer could select to purchase the product at any time between t and the end of the season (inclusive), it would still be optimal for that consumer to continue to adopt the same policy $y(t)$.*

3 The Potential Value of Reward Programs

To study the potential benefits of our proposed surplus-matching reward program (9), we compare the seller's revenue performance to a benchmark case where no reward is offered. In the benchmark case, the seller posts a premium price (p_1) for the first period, and at the end of the season is free to optimally select the best price (p_2). It is such freedom to set the price in the second period that acts to the seller's detriment when consumers are strategic.

Let us first consider all possible equilibria in which all units are sold during the first period (at the price p_1). Obviously, such equilibria can only hold when $\frac{Q}{\lambda} \leq 1$, and they must maintain a consumer response *uniquely* given by[1] $\theta(t) = p_1 e^{\alpha t}$. Consequently, the price p_1 would have to satisfy the condition $\lambda \int_0^1 (1 - p_1 e^{\alpha t})^+ \, dt \geq Q$; in other words, the demand induced by p_1 in the first period must be larger or equal to the quantity Q. Therefore, let p_1^l denotes the *unique* price p_1 that solves the equation $\lambda \int_0^1 (1 - p_1 e^{\alpha t})^+ \, dt = Q$:

$$p_1^l = \begin{cases} \tilde{p}\left(\frac{\alpha Q}{\lambda}\right) \geq e^{-\alpha} & \text{if } 0 \leq \frac{Q}{\lambda} \leq \frac{\alpha - 1 + e^{-\alpha}}{\alpha} \\ \left(1 - \frac{Q}{\lambda}\right) \frac{\alpha}{e^{\alpha} - 1} \leq e^{-\alpha} & \text{if } \frac{\alpha - 1 + e^{-\alpha}}{\alpha} \leq \frac{Q}{\lambda} \leq 1 \end{cases},$$

[1] For clarity and brevity of exposition, we allow the function θ to exceed the value 1. This has no ramifications on the technical correctness of the analysis, since the base valuations of the consumers in the market does not exceed 1.

and note that all (and only) prices in the range $p_1 \in \left[0, p_1^l\right]$ support such equilibrium. The next proposition establishes a *lower bound* on the seller's optimal first-period price.

Proposition 8 *When $\frac{Q}{\lambda} \leq 1$, the retailer's optimal first-period price is larger or equal to p_1^l. Furthermore, within that range, $p_1 = p_1^l$ is the only price for which it is possible to obtain an equilibrium where all units are sold in the first period.*

Next, consider all possible equilibria in which a price $p_1 > p_1^l$ is posted, and the quantity sold in the first period is lower than Q. Obviously, since leftover is expected at the end of the season, the price p_2 would have to be set to a value lower than $\min(p_1, e^{-\alpha})$. Otherwise, it would be optimal for the consumers to follow the strategy $\theta(t) = p_1 e^{\alpha t}$, and furthermore – no mass of consumers would remain at the end of the season with a valuation larger or equal to p_2. This would contradict the optimality of p_2, since by decreasing it even just slightly below $\min(p_1, e^{-\alpha})$, the seller can generate a positive surplus at the end of the season. The next proposition provides an *upper bound* on the seller's optimal first-period price, that can be imposed without loss of optimality.

Proposition 9 *Without loss of optimality, the retailer can restrict his attention to first-period prices lower or equal to p_1^u, defined as follows:*

$$p_1^u \doteq 1 - e^{-\alpha} \min\left(\frac{1}{2}, \frac{Q}{\lambda}\right).$$

Provided such restriction, $p_1 = p_1^u$ is the only price in the range $\left[p_1^l, p_1^u\right]$ (when non-empty) for which the resulting equilibrium is one in which no consumer attempts to purchase in the first period (i.e., $\theta(t) = 1$ for all t).

The latter propositions enable us to consider first period prices ranging from the minimal value p_1^l which results in the sales of all units (or, otherwise, zero price if $Q \geq \lambda$), to the maximal value p_1^u which effectively results in no sales in the first period. For values of $p_1 \in \left(p_1^l, p_1^u\right)$, we provide the following result.

Proposition 10 *When a price $p_1 \in \left(p_1^l, p_1^u\right)$ is posted, and a price $p_2 \leq p_1$ is expected for the second period such that the induced demand will not exceed the available quantity, the consumers follow the purchasing policy $\theta(t) = \max\left(\frac{p_1 - p_2}{e^{-\alpha t} - e^{-\alpha}}, p_1 e^{\alpha t}\right)$.*

This proposition follows straightforward algebraic comparisons of the surpluses associated with an immediate purchase and a delayed purchase; we hence omit the proof. The above results allow us to apply a generic numerical search procedure in which we consider all combinations of the prices p_1 and p_2 that satisfy the conditions of Proposition 10. For each combination, we must indeed verify that p_2 is the best response of the seller to the policy θ; in other words, we have to ensure that p_1, p_2, and θ hold in equilibrium. Given that we have observed only one possible equilibrium for each p_1, we merely select the value of p_1 that yields the largest revenue. Our purpose here is not speed or efficiency, as the full analysis of the benchmark case is tedious and not the central focus of this study.

Table 1 below shows the percentage benefits of our optimal reward program in comparison to the above

	Percentage decline in valuation from beginning to end of season ("degree of fashion")									
Q/λ	**5%**	**15%**	**25%**	**35%**	**45%**	**55%**	**65%**	**75%**	**85%**	**95%**
0.05	0%	1%	1%	2%	2%	2%	3%	4%	4%	6%
0.1	0%	1%	2%	2%	3%	4%	5%	5%	7%	10%
0.15	0%	0%	1%	2%	4%	5%	6%	7%	10%	16%
0.2	0%	0%	1%	2%	4%	5%	7%	10%	13%	21%
0.25	0%	0%	1%	2%	3%	5%	7%	11%	16%	27%
0.3	0%	0%	1%	2%	3%	5%	8%	11%	17%	29%
0.35	0%	0%	1%	2%	3%	5%	8%	12%	18%	31%
0.4	0%	0%	1%	2%	3%	5%	8%	12%	18%	32%
0.45	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.5	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.55	1%	1%	2%	3%	4%	6%	9%	13%	19%	33%
0.6	2%	4%	5%	6%	7%	9%	12%	15%	21%	33%
0.65	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%
0.7	2%	5%	8%	9%	10%	11%	13%	16%	21%	33%
0.75	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%
0.8	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%
0.85	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%
0.9	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%
0.95	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%
1	2%	5%	8%	9%	10%	11%	13%	16%	21%	34%

Table 1: The percentage benefits of optimal reward program (benefits over a dynamic two-price strategy).

benchmark case. The rows on the table display different levels of inventory, measured by Q/λ. The columns display different levels of the "degree of fashion" measure $\delta = 1 - e^{-\alpha}$, which represent the decline in valuations from the beginning of the season (at time $t = 0$) to its end. Note that the benefits are all non-negative, a fact rigorously established in the following theorem.

Theorem 3 *The performance of the optimal surplus-matching reward program is at least as good as that of the benchmark two-price strategy.*

The table suggests that the benefits of our optimal reward program are indeed particularly useful when dealing with fashion-like products, in which consumers experience a considerably high decline in valuation if they decide to delay their purchase. Additionally, and consistent with existing research on strategic consumer behavior, these benefits seem to increase substantially with the level of inventory (when strategic waiting is intense in the benchmark model). For example, even when the degree of fashion is relatively modest (say $5\%-25\%$), the benefits can reach levels of up to $2\%-8\%$; indeed, even such a single-digit percent of increase in *revenue* performance can be highly significant for sellers.

4 Comparison to Alternative Pricing Strategies

As discussed in the introduction, the literature on strategic consumer behavior has studied two pricing strategies for mitigating the adverse impact of this phenomenon: price commitment (PC), and inter-temporal price matching (PM) strategies. In this section, we briefly present a model for each one of these mechanisms, provides some theoretical results, and present and discuss the differences we observe in performance in comparison to our optimal surplus-matching reward program. But, first, note that each one of the two mechanisms can be described as a special case of a reward program: for example, a PC strategy can be described as a reward program that offers consumers a reward of $r(t, p_1, p_2) = M \cdot \left|p_2 - p_2^0\right|$ for some very large value M, and for any arbitrary p_2^0 announced in advance. This removes any incentive from the seller to deviate from the price path $\left(p_1, p_2^0\right)$. Offering a PM plan using our general reward program structure is even simpler, as it trivially means: $r(t, p_1, p_2) = \max(p_1 - p_2, 0)$. We conclude,

Theorem 4 *The performance of the optimal surplus-matching reward program is at least as good as that of a price commitment and an inter-temporal price matching strategies.*

(Proof is straightforward.) While it is easy to see that both PM and PC can be presented as special cases of the class of early-purchase reward programs, we cannot utilize the analysis of the previous section to compute their revenue performance, since neither PC nor PM maintain the surplus-matching property. We therefore discuss the computational procedures for the two pricing strategies in the dedicated sub-sections below.

4.1 Inter-Temporal Price-Matching (PM) Strategies

Inter-temporal PM strategies, in which a seller promises to refund the difference between the price paid by a consumer and a discounted price (if offered in the second period), have been discussed and analyzed in the literature on strategic consumer behavior (see, e.g., Lai et al., 2010). The rationale behind such offering is similar to what we discussed above: a commitment to refund consumers in case of a discount serves as a mechanism to ensure that no significant markdowns will be offered at the end of the season, discouraging consumers from postponing their purchases.

To compute the optimal revenue that can be obtained under a PM strategy, we first identify the optimal premium price and its corresponding subgame equilibrium.

Proposition 11 *Suppose that a seller adopts a PM strategy, offering to issue a refund of* $(p_1 - p_2)^+$ *to any consumer who purchases at the premium price. Also, define* $\bar{\alpha}$ *as the unique solution to the equation* $1 - \frac{1}{2}x - e^{-x} = 0$ *(i.e.,* $\bar{\alpha} \approx 1.59$*, representing a setting where valuations decline close to* $\delta = 80\%$ *from the beginning of the season to the end of the season). Then, it is optimal for the seller to set the premium price (*p_1^{PM}*) as follows:*

(i) *If* $\alpha \leq \bar{\alpha}$*,*

$$p_1^{PM} \doteq \begin{cases} \frac{1}{2}\frac{\alpha}{e^{\alpha}-1} & Q \geq \frac{\lambda}{2} \\ \left(1 - \frac{Q}{\lambda}\right)\frac{\alpha}{e^{\alpha}-1} & \lambda\left(1 - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha}\right) \leq Q < \frac{\lambda}{2} \\ \tilde{p}\left(\alpha\frac{Q}{\lambda}\right) & Q < \lambda\left(1 - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha}\right) \end{cases}$$

where $\tilde{p}(x)$ *is the unique solution to the equation* $\tilde{p} - \ln(\tilde{p}) = 1 + x$*.*

(ii) *If* $\alpha > \bar{\alpha}$ *(very high "degree of fashion"),*

$$p_1^{PM} \doteq \begin{cases} e^{-\bar{\alpha}} & Q \geq \frac{\lambda}{2} \cdot \frac{\bar{\alpha}}{\alpha} \\ \tilde{p}\left(\alpha\frac{Q}{\lambda}\right) & Q < \frac{\lambda}{2} \cdot \frac{\bar{\alpha}}{\alpha} \end{cases}$$

Consequently, in the subgame, consumers arriving at time t *will purchase the product immediately if their base valuations are higher than* $p_1^{PM} e^{\alpha t}$*, and the seller will not change the price at the end of the season (i.e.,* $p_2^{PM} = p_1^{PM}$*).*

Using the above result, we can easily proceed to calculate the optimal revenue for a PM strategy.

Proposition 12 *The seller's optimal revenue under a price matching strategy is given by:*

(i) *If* $\alpha \leq \bar{\alpha}$*,*

$$\Pi^{PM}(Q) = \begin{cases} \frac{\lambda}{4}\frac{\alpha}{e^{\alpha}-1} & Q \geq \frac{\lambda}{2} \\ Q \cdot \left(1 - \frac{Q}{\lambda}\right)\frac{\alpha}{e^{\alpha}-1} & \lambda\left(1 - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha}\right) \leq Q < \frac{\lambda}{2} \\ Q \cdot \tilde{p}\left(\alpha\frac{Q}{\lambda}\right) & Q < \lambda\left(1 - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha}\right) \end{cases}$$

(ii) *If $\alpha > \bar{\alpha}$ (very high "degree of fashion"),*

$$\Pi^{PM}(Q) = \begin{cases} \frac{\lambda}{2} e^{-\bar{\alpha}} \cdot \frac{\bar{\alpha}}{\alpha} & Q \geq \frac{\lambda}{2} \cdot \frac{\bar{\alpha}}{\alpha} \\ Q \cdot \tilde{p}\left(\alpha \frac{Q}{\lambda}\right) & Q < \frac{\lambda}{2} \cdot \frac{\bar{\alpha}}{\alpha} \end{cases},$$

where $\bar{\alpha}$ and $\tilde{p}$ are as defined in Proposition 11.

Table 2 below shows the percentage benefits of our optimal reward program in comparison to an optimal PM strategy. The table's layout is similar to that of Table 1 in Section 3, and demonstrates the following

	Percentage decline in valuation from beginning to end of season ("degree of fashion")									
Q/λ	**5%**	**15%**	**25%**	**35%**	**45%**	**55%**	**65%**	**75%**	**85%**	**95%**
0.05	0%	1%	1%	1%	2%	2%	3%	3%	4%	6%
0.1	0%	1%	2%	2%	3%	3%	4%	5%	7%	10%
0.15	0%	0%	1%	3%	4%	5%	6%	7%	10%	15%
0.2	0%	0%	1%	2%	4%	6%	7%	10%	13%	21%
0.25	0%	0%	1%	2%	4%	6%	9%	12%	16%	28%
0.3	0%	0%	1%	2%	4%	6%	10%	14%	20%	35%
0.35	0%	0%	1%	2%	3%	6%	10%	16%	24%	40%
0.4	0%	0%	1%	2%	3%	6%	10%	17%	27%	44%
0.45	0%	0%	1%	2%	3%	5%	10%	17%	30%	46%
0.5	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.55	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.6	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.65	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.7	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.75	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.8	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.85	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.9	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
0.95	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%
1	0%	0%	1%	2%	3%	5%	10%	17%	31%	47%

Table 2: The percentage advantage of optimal reward programs in comparison to optimal price matching strategies.

pattern. When the product sold is not "fashion-like", the benefits of offering a reward program instead of a price-matching program diminish. However, the potential advantage of a surplus-matching reward program becomes significant (i.e., $\geq 2\%$), for almost *all* levels of inventory, as we move our attention to markets with higher degrees of fashion (about $\delta \geq 35\%$). In light of our detailed discussions of the optimal surplus-matching reward plan (9), the numbers observed on the table are not surprising. For a PM strategy to be effective, it must provide the seller with the flexibility to drop prices – hence exercising price discrimination – but also discourage the seller from offering significant price reductions. However, when the degree of fashion is high, and the seller needs to drop prices significantly in order to generate additional revenues at the end of the season, a PM plan can be highly counterproductive. In fact, note from the two right columns of

Tables 1 and 2 that a PM plan's performance can be even worse than that of the benchmark case. Our surplus-matching reward program resolve this shortcoming of PM strategies by "correcting" (reducing) the payback to the consumers in case of a price discount. As we argued, this is done by optimally accounting for the fact that consumers that buy the product earlier in the season enjoy it more than if they purchase it at its end. Therefore, earlier in the season, consumers do not need as large of a price-matching guarantee than they would need at a later time, in order to be motivated to purchase the product at the premium price.

4.2 Price Commitment (PC) Strategies

The potential value of PC strategies, studied in our literature (see, e.g., Aviv and Pazgal, 2008), lies in the fact that the seller can credibly commit to not offer significant discounts. Obviously, this limits the seller's ability to exercise price discrimination, but it also mitigates the adverse effect of strategic waiting due to the fact that rational expectations for significant price drops can be avoided.

The computation of the revenue performance under optimal PC strategies is technically simplified due to the commitment power. In other words, we no longer need to require that p_2 is indeed the seller's best response at the end of the season. However, a different complication now arises, as under an optimal PC strategy, the seller may induce *shortage risk*. Using such approach, the seller may deliberately commit to a sufficiently large markdown at the end of the season, to induce more demand than the leftover inventory. Thus, creating a rationing risk in order to entice high-valuation consumers to purchase early in the season at the premium price; see, e.g., Liu and Van Ryzin (2008).

Following the same arguments made in §3, we can verify that without loss of optimality, the seller can continue to focus his attention to pairs of prices (p_1, p_2) such that $p_1 \geq p_1^l$ and $p_2 \leq p_1$. Using (2) and (3), we obtain the following result.

Proposition 13 *Consider any posted pair of prices (p_1, p_2), where $p_1 \geq p_1^l$ and $p_2 \leq p_1$. Also, let β be the allocation probability resulting from the consumers' purchasing activity. Then, it is optimal for the consumers to adopt the purchasing policy $\theta(t) = \max\left(\frac{p_1 - \beta \cdot p_2}{e^{-\alpha t} - \beta \cdot e^{-\alpha}}, p_1 e^{\alpha t}\right)$. Furthermore, β is unique for (p_1, p_2).*

Utilizing the above proposition, we adopt the following computational procedure. For any given pair of prices (p_1, p_2) that satisfies the conditions $p_1 \geq p_1^l$ and $p_2 \leq p_1$, we calculate the unique consumers' response function θ. In fact, the proof of this proposition is constructed in a way that also provides a simple and efficient procedure to identify the unique value of β for each and every pair of prices that we examine. We then compute the revenue performance via

$$p_2 \cdot \min\left(\lambda \int_0^1 \left(1 - \min\left(p_1 e^{\alpha t}, p_2 e^{\alpha}\right)\right)^+ dt, Q\right) + (p_1 - p_2)\, \lambda \int_0^1 (1 - \theta(t))^+ \, dt$$

Table 3 below shows the percentage benefits of our optimal reward program in comparison to an optimal PC strategy. The results are identical in pattern to those observed in our comparison to PM strategies; see

	Percentage decline in valuation from beginning to end of season ("degree of fashion")									
Q/λ	**5%**	**15%**	**25%**	**35%**	**45%**	**55%**	**65%**	**75%**	**85%**	**95%**
0.05	0%	1%	1%	1%	2%	2%	3%	3%	4%	6%
0.1	0%	1%	2%	2%	3%	3%	4%	5%	7%	10%
0.15	0%	0%	1%	2%	4%	5%	6%	7%	10%	15%
0.2	0%	0%	1%	2%	4%	5%	7%	9%	13%	21%
0.25	0%	0%	1%	2%	3%	5%	7%	11%	15%	26%
0.3	0%	0%	1%	2%	3%	5%	8%	11%	17%	29%
0.35	0%	0%	1%	2%	3%	5%	8%	12%	18%	31%
0.4	0%	0%	1%	2%	3%	5%	8%	12%	18%	32%
0.45	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.5	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.55	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.6	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.65	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.7	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.75	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.8	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.85	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.9	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
0.95	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%
1	0%	0%	1%	2%	3%	5%	8%	12%	19%	33%

Table 3: The percentage advantage of optimal reward programs in comparison to optimal price commitment strategies.

Table 2: When selling "fashion-like" products, the seller cannot effectively use price commitment strategies, since for them to be effective in exercising price discrimination, the seller would have to commit to a low price. But that would lead to significant strategic waiting.

5 On the Robustness of Early–Purchase Reward Programs: An Illustration via the Model of Lai et al. (2010)

Despite our primary focus on providing fundamental theory in this research paper, we devote the current section to provide an *illustration* of the potential value of early-purchase reward programs, by applying our results in an alternative model of a market consisting strategic consumers. The paper by Lai et al. (2010), mentioned in our introduction, provides a comprehensive analysis of the potential benefits of inter-temporal price matching policies (titled "posterior price matching" therein) under a few broader conditions than those to which we have confined ourselves. In their fundamental model, the seller is faced with a market that consist of low-end consumers (many of them; all share a common valuation V_L, which falls below the seller's per-unit cost), and high-end consumers (all share the same valuation V_H). The size of the latter group (λ) is uncertain. Among the high-end consumers, a portion ϕ are strategic in the same sense considered in our paper, and the rest are myopic. The seller is interested in maximizing his profit by first determining the initial price for the "first period" (p_1). Then, the seller and the consumers are engaged in a game in which the seller sets the quantity to be brought to the store (Q), and the consumers determine their purchasing rule (a probability q for buying in the first period at the posted price). Eventually, the seller sets the price at the beginning of the second period (p_2). A key market characteristic is that the high-end consumers' valuations decline from V_H to V_h from the first to the second period. For the purpose of our discussion, we shall consider the fraction $(V_H - V_h)/V_H$ as the "degree of fashion." Production cost and inventory carrying costs are considered in the profit model. Two scenarios are analyzed: one, a benchmark case, in which the seller is free to select the prices optimally at the beginning of each period. In particular, the seller does not possess any commitment power. This model is analogically similar to our model of §3. Second, is a case in which the seller offers a price matching guarantee to its consumers, as we considered in §4.1. It is assumed that all strategic consumers, but only a portion γ of the myopic consumers, who buy in the first period end up requesting the price match (if there is a drop in price).

To provide an illustration of the robustness of our surplus-matching reward program, we have replicated the analysis conducted in Lai et al. (2010) for a degree of fashion $(V_H - V_h)/V_H \in \{20\%, 50\%\}$ and a percentage of strategic consumers among the high-end segment $\phi \in \{25\%, 50\%, 75\%\}$. We have also limited our attention to settings in which the seller faces a modest-to-high degree of uncertainty regarding the high-end segment size λ (specifically, we focused on the case of $\sigma/\mu \in \{0.5, 1\}$). Modest-to-high degrees of fashion, and relatively large market size uncertainties, are the conditions under which we believe that reward programs could be particularly beneficial.

As a candidate for a reward program, we propose a *simple* heuristic procedure in which the seller offers the following plan:

$$r(p_1, p_2) = \left(p_1 - V_H + (V_h - p_2)^+\right)^+ \tag{10}$$

The rationale behind (10) is that the seller targets the high-end consumers by charging them an effective price equal to their valuation V_H (in fact, this can be done by merely charging $p_1 = V_H$). Additionally, the seller promises to match the price to p_2 in the second period, but only up to the high-end consumers' valuation at that time (i.e., V_h). Note that our purpose is not to search for the best possible reward program. Instead, we aim to demonstrate the simplicity of implementing our qualitative insights (regarding the structure of the optimal reward function) in a different setting – that of Lai et al. (2010). Therefore, the results we report below represent conservative estimates of the potential value of optimal reward programs in that particular setting. To analyze the performance of the reward program (10), we applied a generic dynamic programming procedure, by first identifying the optimal second-period price under the reward plan (10). We then utilized an iterative algorithm to compute the equilibrium in the game between the consumers (setting the value of q) and the seller (selecting the optimal quantity Q), and followed with a search for the optimal first-period price.

Table 4 below shows the average percentage advantage of our heuristic early-purchase reward program (10) in comparison to a posterior price matching plan, across the parameter values specified above. We

Percentage of strategic consumers (ϕ)	Degree of fashion (% decline in valuation)			
	20%		50%	
	cv = 0.5	cv = 1	cv = 0.5	cv = 1
25%	0.4%	1.9%	2.2%	6.8%
50%	0.2%	1.8%	1.5%	6.6%
75%	0.1%	1.6%	1.0%	6.2%

Table 4: The average percentage advantage of a simple (heuristic) early-purchase reward program over the optimal price commitment strategy in the model setting of Lai et al. (2010). The table is based on the following parameter combinations: $\mathrm{E}[\lambda] = 100$, $V_H = 10$, $V_L = 2$, $\gamma = 0.25$, $\rho = 0.04$. Averages are taken across the three cost values $c \in \{4, 6, 8\}$. The value of cv represents the coefficient of variation $\frac{\sigma}{\mu}$.

note that, not surprisingly, under the parameter combinations considered above, both programs significantly dominate the benchmark setting in which the seller does not offer any plan. As can be seen from the table, the advantage of using an early-purchase reward program grows with the degree of fashion. As we explained in §4.1, we expect the numbers to be even much higher when the degree of fashion is further increased. As can be noticed, the percentage of strategic consumers in the high-end segment does not influence the potential advantage of early-purchase reward programs in any significant way. The slight decrease in the advantage as a function of ϕ can be explained by the fact that the reward program issues a lower price match payment (than a posterior price-matching plan) to myopic consumers, for whom such payments are not influential in the buy-now versus buy-later decision. Finally, note that the uncertainty regarding the high-end segment size greatly matters. Our interpretation of this observation requires a deeper insight into the fundamental value of early-purchase reward programs. We conjecture from our theoretical and numerical analyses above, that reward programs draw their benefits from enhanced *segmentation* ability; i.e., in providing the seller with the means to charge the right price from the right customer or segment. Since in Lai et al.'s model the valuation is fixed across all high-end customers, the power of an early-purchase reward program emerges from the seller's ability to attract one segment versus the other. This is the reason that the level of uncertainty about the segment size can be managed better via the early-purchase program (10).

6 Conclusion

This paper proposes a scientific model in which a seller of a seasonal fashion good is challenged with setting the optimal price for its product in a market consisting of strategic consumers. A variety of earlier papers on this subject have studied the adverse consequences of strategic consumer behavior. In particular, a couple of mechanisms such as price-commitment strategies and inter-temporal price-matching guarantees have been proposed as possible ways to counteract this phenomenon. We essentially focus on a single, yet thought-provoking question that stems from the following logic. Let us suppose that the seller could offer an early-purchase reward (EPR) program to which all consumers who buy the product at premium price would be automatically enrolled. Under the program, the seller is committed to issue a credit to the consumer as a function of the time of purchase, the sales realization during the season, and the price of the product at the end of the season. Price-commitment strategies and price-matching guarantees are thus special cases in the very broad class of EPR programs. The following question naturally arises: what is the structure of the optimal EPR program, and what is its potential value? In this vein, our approach is prescriptive, rather than normative, and our interest is primarily in delivering managerial insights into the choice of innovative EPR-based pricing policies.

By confining ourselves to a variety of simplifying assumptions regarding the market structure and dynamics, we gained the ability to obtain a complete analytical characterization of the optimal reward program that a seller can offer to its consumers. The structure of such a reward program reveals an interesting finding. First, the seller should offer a non-contingent refund that can be perceived as a "participation bonus". This results in the consumers paying an effective time-dependent price that attracts a particular (planned) segment of the market. The second component is contingent on possible price markdowns, which can be viewed as a "modified price matching guarantee". Here, instead of matching the effective price paid by the consumer to the price listed at the end of the season, the seller reduces such payment to reflect the fact that the consumer's valuation for the product declines over the course of the season. In this sense, the reward program utilizes a logically-fair mechanism that – on one hand – protects the consumer against price drops, but – on the other hand – charges the consumers for the fact that they gain a larger value by consuming the product earlier rather than later (at the end of the season).

To gauge the magnitude of the potential gains that can be achieved by implementing EPR programs, we have conducted a numerical comparison between the performances of EPR, price-commitment, and price-matching strategies. One of the key takeaways from this study is that optimal reward programs increase substantially in value as the level of inventory increases and as the product becomes more fashion-like, in the sense that consumers gain significantly higher utility from consuming it earlier rather than later in the season. Even when the level of inventory is low and the degree of fashion is modest, the benefits of implementing an EPR program are likely to be significant.

Finally, the rich modelling framework of Lai et al. (2010), which includes market size uncertainty, cost considerations, mixture of strategic and non-strategic consumers, and more, presented to us a valuable opportunity to further illustrate the potential value of EPR programs. While Lai et al.'s paper is focused on the study of the benefits of posterior price-matching guarantees, we found their model readily available for evaluating the performance of a simple (heuristic) EPR program. The comparison between the two mechanisms – price-matching vs. EPR – led us to conjecture that EPR programs can be beneficial in settings involving modest-to-high degrees of fashion and high degrees of market size uncertainty, regardless of the percentage of strategic consumers in the market. Of course, in order to ensure a selection of an appropriate EPR program, the seller must be able to gauge the extent to which strategic consumer behavior prevails; see, e.g., Li et al. (2013) for a related empirical study based on data gathered in the air-travel industry.

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Appendix: Proofs

Proof of Proposition 1. Let us use the expression "focal consumer" to describe the particular consumer considered in this proposition. Let's denote this focal consumer's surplus from an immediate purchase as $s_1(v,t) \doteq 1\{\mathcal{A}_1\}\cdot(ve^{-\alpha t}-p_1+r(t,p_1,p_2(I,R),I))$, where $\mathcal{A}_1$ is the event under which this focal consumer is actually able to obtain this product at time t and $1\{\cdot\}$ is a indicator function. Similarly, we denote the surplus associated with a wait decision as $s_2(v) \doteq 1\{\mathcal{A}_2\}\cdot(ve^{-\alpha}-p_2(I,R))^+$. It is possible that $\mathcal{A}_1$, $\mathcal{A}_2$, and I are correlated with each other, but they all are independent with this focal consumer's valuation v. Notice that $\mathrm{E}_{\mathcal{A}_1,I}[s_1(v,t)]$ and $\mathrm{E}_{\mathcal{A}_2,I}[s_2(v)]$ are continuous functions of v, and $\mathrm{E}_{\mathcal{A}_2,I}[s_2(0)] \geq 0 > \mathrm{E}_{\mathcal{A}_1,I}[s_1(0,t)]$. Thus, there are two possible cases: (i) if the two functions never cross for v in the range $[0,1]$, then it is optimal for the focal consumer to wait. Equivalently, this focal consumer will adopt a threshold policy with $\theta(t)=1$; (ii) if the two functions meet at a given point θ, i.e.,

$$\mathrm{E}_{\mathcal{A}_1,I}[s_1(\theta,t)] = \mathrm{E}_{\mathcal{A}_2,I}[s_2(\theta)], \tag{11}$$

then we will show that this crossing point is unique. As

$$\frac{\partial}{\partial v}\mathrm{E}_{\mathcal{A}_1,I}\left[s_1\left(v,t\right)\right]=\left(e^{-\alpha t}\right)P\left[\mathcal{A}_1\right]>\left(e^{-\alpha}\right)P\left[\mathcal{A}_2\right]\geq\frac{\partial}{\partial v}\mathrm{E}_{\mathcal{A}_2,I}\left[s_2\left(v\right)\right]$$

for $t\in[0,1)$, (11) has at most one solution, which establishes the uniqueness of θ. Therefore, it is optimal for the focal consumer to purchase a unit during the main season if $v>\theta$, and wait if $v\leq\theta$.

Proof of Proposition 2. Given all other consumers' purchasing threshold function Θ, we first denote an individual consumer's expected surplus from an immediate purchase and that from a wait decision as $\mathrm{E}_{\mathcal{A}_1,I}\left[\tilde{s}_1\left(v,t|\Theta\right)\right]$ and $\mathrm{E}_{\mathcal{A}_2,I}\left[\tilde{s}_2\left(v|\Theta\right)\right]$ respectively. As shown in the Proposition 1, this individual consumer's decision is of threshold type with a unique value $\theta^*\left(t|\Theta\right)$: $\mathrm{E}_{\mathcal{A}_1,I}\left[\tilde{s}_1\left(v,t|\Theta\right)\right]>\mathrm{E}_{\mathcal{A}_2,I}\left[\tilde{s}_2\left(v|\Theta\right)\right]$ for all $v>\theta^*\left(t|\Theta\right)$, and $\mathrm{E}_{\mathcal{A}_1,I}\left[\tilde{s}_1\left(v,t|\Theta\right)\right]\leq\mathrm{E}_{\mathcal{A}_2,I}\left[\tilde{s}_2\left(v|\Theta\right)\right]$ for all $v\leq\theta^*\left(t|\Theta\right)$. Therefore, the equilibrium in the consumers' game will be given by the fixed point of $\theta^*\left(t|\Theta\right)$ with respect to $\theta\left(t\right)$. And the existence of such fixed point follows directly from the continuity of $\theta^*\left(t|\Theta\right)$ with respect to $\theta\left(t\right)$ (via the Implicit Function Theorem), and the Brouwer's Fixed-point Theorem.

Proof of Theorem 1. First, note that utilizing the proof of Proposition 1, the necessary and sufficient condition for Equation 4 to hold is $s_1\left(1,t|\Theta,R\right)\geq s_2\left(1|\Theta,R\right)$. Let's suppose $\tilde{r}\left(t,p_1,p_2\left(\Theta,\tilde{R}\right),I\left(\Theta\right)\right)$ is the optimal reward program and $s_1\left(1,t|\Theta,\tilde{R}\right)<s_2\left(1|\Theta,\tilde{R}\right)$. Next, we will show that it will not affect the seller's optimality, if the seller chooses to increase its rewards from $\tilde{r}\left(t,p_1,p_2,I\right)$ to $r\left(t,p_1,p_2,I\right)$ by $\left(s_2\left(1|\Theta,\tilde{R}\right)-s_1\left(1,t|\Theta,\tilde{R}\right)\right)$ amount. Note that under the new reward program, consumers' purchasing behavior will remain unchanged, as under both programs all consumers arrived at time t wait for the end of the season. Also, the seller's revenue performance and the pricing decision at the end of the season will remain unchanged under the new reward program, as no consumer purchases at time t under both reward programs. Therefore, it is still optimal for the seller to adopt the new reward program $r\left(t,p_1,p_2,I\right)$ under which $s_1\left(1,t|\Theta,R\right)=s_2\left(1|\Theta,R\right)$ holds. In other words, without loss of optimality, the seller can focus its attention on those $r\left(t,p_1,p_2,I\right)$, under which equation 4 always holds.

Proof of Proposition 5. First, observe that we can focus on values of $p_2\leq\theta\left(1\right)e^{-\alpha}$. To see this, note that both (6) and (7) are constant for all $p_2\geq\theta\left(1\right)e^{-\alpha}$. Now, let us consider two cases: (i) When $\lambda\left(1-\theta\left(0\right)\right)<Q$, the value of p_2 must be in the range $\Omega_{(i)}\doteq\left[\left(1-\frac{Q}{\lambda}\right)^+e^{-\alpha},\theta\left(1\right)e^{-\alpha}\right]$, where (7) is satisfied. Let $\theta^{-1}\left(v\right)$ be the first time t at which $\theta\left(t\right)=v$, with the understanding that $\theta^{-1}\left(v\right)=0$ for all $v\in[\left(1-\frac{Q}{\lambda}\right)^+,\theta\left(0\right))$. (Note that θ^{-1} is a non-decreasing, right-continuous, and left-differentiable function).

Then,

$$
\begin{aligned}
& \max_{\text{s.t. } (7)} \{\text{Eq. } (6)\} \\
= \;& \lambda \cdot \max_{p_2 \in \Omega_{(i)}} \left\{ \int_0^1 \left(\theta(t) e^{-\alpha t} - \left(\theta(t) e^{-\alpha} - p_2\right)^+ \right) \cdot (1 - \theta(t)) \, dt + \int_0^1 p_2 \cdot \left(\theta(t) - p_2 e^{\alpha}\right)^+ dt \right\} \\
= \;& \lambda \int_0^1 e^{-\alpha t} \cdot \theta(t) \cdot (1 - \theta(t)) \, dt \\
& + \lambda \cdot \max_{p_2 \in \Omega_{(i)}} \left\{ \int_{\theta^{-1}(p_2 e^{\alpha})}^1 \left(\theta(t) - p_2 e^{\alpha}\right) \cdot \left(p_2 - e^{-\alpha}(1 - \theta(t))\right) dt \right\}
\end{aligned}
$$

Note that by taking the derivative of the expression within the maximum operation, at points where $\theta^{-1}(p_2 e^{\alpha})$ is right-differentiable, we obtain $\frac{\partial}{\partial p_2}(\ldots) = (1 - 2p_2 e^{\alpha})\left(1 - \theta^{-1}(p_2 e^{\alpha})\right)$. Additionally, at points in which $\theta^{-1}(p_2 e^{\alpha})$ jumps upward, the revenue expression remains continuous. Therefore, $p_2 = \frac{1}{2} e^{-\alpha}$ is optimal for the unconstrained maximization problem, which yields the solution:

$$p_2^* = \min \left\{ \max \left\{ \frac{1}{2} e^{-\alpha}, \left(1 - \frac{Q}{\lambda}\right)^+ e^{-\alpha} \right\}, \theta(1) e^{-\alpha} \right\}.$$

(ii) When $\lambda(1 - \theta(0)) \geq Q$, the value of p_2 must be in the range $\Omega_{(ii)} \doteq [\tilde{p}_2, \theta(1) e^{-\alpha}]$, where $\tilde{p}_2$ is defined as the unique value of p_2 that satisfies constraint (7) with an equality. A similar analysis to that of case (i) yields the optimal price:

$$p_2^* = \min \left\{ \max \left\{ \frac{1}{2} e^{-\alpha}, \tilde{p}_2 \right\}, \theta(1) e^{-\alpha} \right\}.$$

Finally, note that in both cases the value of p_2^* will be equal to $\theta(1) e^{-\alpha}$ if and only if $\theta(1) < \frac{1}{2}$.

Proof of Proposition 6. First, we establish that it is optimal to focus on $\theta(1) \geq \frac{1}{2}$. In contrast, suppose that there was an optimal Θ with $\theta(1) < \frac{1}{2}$. Then, by the proof of Proposition 5, $p_2^* = \theta(1) e^{-\alpha}$, and the seller's revenue is given by $\lambda \int_0^1 e^{-\alpha t} \cdot \theta(t) \cdot (1 - \theta(t)) \, dt$. Furthermore, the constraint (7) would imply that $\lambda \int_0^1 \left(1 - \frac{1}{2}\right) dt \leq \lambda \int_0^1 (1 - \theta(t)) \, dt \leq Q$. This means that the alternative segmentation plan $\tilde{\Theta} \equiv \frac{1}{2}$ with $\tilde{p}_2 = \frac{1}{2} e^{-\alpha}$ would bring strictly better revenues (as it point-wise maximizes the expression within the integral), and continues to satisfy equation (7). Hence, a contradiction. Next, we show that $\theta(0) \geq \frac{1}{2}$. Suppose not; then, by Proposition 5 and the fact that $\theta(1) \geq \frac{1}{2}$, we get $p_2^* \geq \frac{1}{2} e^{-\alpha}$. This means that by replacing Θ with $\tilde{\Theta} \doteq \max\left(\Theta, \frac{1}{2}\right)$ – even if we keep the same p_2^* – we increase the revenue by $\lambda \int_0^{\theta^{-1}\left(\frac{1}{2}\right)} e^{-\alpha t} \cdot \left[\frac{1}{4} - \theta(t) \cdot (1 - \theta(t))\right] dt > 0$ (observe that $\tilde{\Theta}$ uses less inventory than Θ, and hence it is feasible). We are now ready to establish our proposition. Hereafter, we shall use the notation Θ^* and p_2^* to denote the optimal segmentation. We distinguish between two cases:

(i) $Q \geq \frac{\lambda}{2}$. Here, since we have established that $\Theta^* \geq \frac{1}{2}$ and since $1 - \frac{Q}{\lambda} \leq \frac{1}{2}$, the first case in (8) applies, and we get $p_2^* = \frac{1}{2} e^{-\alpha}$. If $\theta^*(0) > \frac{1}{2}$, we obtain the revenue expression $\lambda \int_0^1 e^{-\alpha t} \cdot \theta(t) \cdot (1 - \theta(t))\, dt$, which contradicts the optimality of Θ^*, as $\tilde{\Theta} \equiv \theta^*(0)$ would maximize this expression, but with inferior performance to $\tilde{\Theta} \equiv \frac{1}{2}$. Thus, we must have $\theta^*(0) = \frac{1}{2}$. But in this case, the revenue expression is given by $\lambda \int_0^1 \left[e^{-\alpha t} \theta(t) \cdot (1 - \theta(t)) + e^{-\alpha} \left(\theta(t) - \frac{1}{2} \right)^2 \right] dt$, which is point-wise maximized with $\Theta^* \equiv \frac{1}{2}$.

(ii) and (iii) $Q < \frac{\lambda}{2}$. Suppose that $\theta^*(0) > 1 - \frac{Q}{\lambda} > \frac{1}{2}$. Then, $p_2^* = \left(1 - \frac{Q}{\lambda} \right) e^{-\alpha}$ by Proposition 5, and the revenue function is given by $\lambda \int_0^1 \left[e^{-\alpha t} \theta(t) \cdot (1 - \theta(t)) + e^{-\alpha} \left(\theta(t) - p_2^* e^{\alpha} \right) \left(p_2^* e^{\alpha} - 1 + \theta(t) \right) \right] dt$. But this means that it would be optimal and feasible to replace Θ^* by a better policy $\tilde{\Theta} \equiv 1 - \frac{Q}{\lambda}$; thus – a contradiction. Therefore, we must have $\frac{1}{2} \leq \theta^*(0) \leq 1 - \frac{Q}{\lambda}$. Furthermore, in view of the proof of Proposition 5, the constraint (7) is binding, and it is easy to verify that the optimal end-of-season price also satisfies: $p_2^*(\Theta^*) = \theta^*(1) e^{-\alpha}$, meaning that no customer purchases at the end of the season. Thus, the revenue optimization is given by:

$$\begin{aligned} &\max_{\Theta} \lambda \int_0^1 e^{-\alpha t} \cdot \theta(t) \cdot (1 - \theta(t))\, dt \\ &\text{s.t. } \lambda \int_0^1 (1 - \theta(t))\, dt = Q \end{aligned}$$

The above problem is simple, as it can be point-wise optimized (over $\theta(t)$'s) using the Lagrangian method; we omit the details.

Proof of Theorem 2. Let us first examine the consumers' response to any given anticipated price $p_2 \in [0, e^{-\alpha}]$. By (2) and (3), we need to compare the values $s_1(v) = v e^{-\alpha t} + (y(t) \cdot e^{-\alpha} - p_2)^+ - y(t) \cdot e^{-\alpha t}$ and $s_2(v) = (v e^{-\alpha} - p_2)^+$. Two cases arise: (i) $p_2 < y(t) \cdot e^{-\alpha}$: here, it is easy to verify that $s_1(p_2 e^{\alpha}) = (e^{-\alpha} - e^{-\alpha t}) \cdot (y(t) - p_2 e^{\alpha}) < 0 = s_2(p_2 e^{\alpha})$ for all $t \in [0, 1)$, while $s_1(1) - s_2(1) = (e^{-\alpha t} - e^{-\alpha})(1 - y(t)) > 0$. Consequently, since $\partial s_1 / \partial v > \partial s_2 / \partial v$, we obtain $\theta(t) = y(t)$ – the unique solution to $\theta(t) e^{-\alpha t} + y(t) \cdot e^{-\alpha} - p_2 - y(t) \cdot e^{-\alpha t} = \theta(t) e^{-\alpha} - p_2$; (ii) $p_2 \geq y(t) \cdot e^{-\alpha}$: here, $s_1(p_2 e^{\alpha}) = e^{-\alpha t} \cdot (p_2 e^{\alpha} - y(t)) \geq 0 = s_2(p_2 e^{\alpha})$, and we again obtain $\theta(t) = y(t)$, this time – the unique solution to $s_1(\theta(t)) = \theta(t) e^{-\alpha t} - y(t) \cdot e^{-\alpha t} = 0$. We now identify the optimal solution to (1) with R as given in the Theorem. But, first, recall that it is not optimal to set a price p_2 that induces a demand that is larger than the leftover inventory; see Proposition 3. Rewriting the optimization expression for π_2, and plugging in the value $\theta(t) = y(t)$, we obtain the equivalent problem: $\max_{p_2} \left\{ \int_0^1 (y(t) e^{-\alpha} - p_2)^+ (p_2 e^{\alpha} - 1 + y(t))\, dt \right\}$, for which the solution is $p_2 = \frac{1}{2} \cdot e^{-\alpha}$ (as it point-wise maximizes the expression within the integral). However, in view of the constraint (7) which is active when $Q < \lambda/2$, we must select $p_2 = y(1) e^{-\alpha}$, at which the constraint holds with equality.

The expressions for the seller's revenue follow by trivial algebra. Finally, we establish the range of p_1 by ensuring that the reward is non-negative for any possible price p_2; i.e., the minimum level of p_1 is set to $\max_t \{y(t) e^{-\alpha t}\} = y(0)$.

Proof of Proposition 7. Consider any focal consumer with base valuation v, arriving at time $t < 1$. Then, for any $\tilde{t} \in (t, 1]$ and $v \geq y(t)$ we have: $ve^{-\alpha t} - y(t) e^{-\alpha t} \geq \left(v - y\left(\tilde{t}\right)\right) e^{-\alpha t} \geq \left(v - y\left(\tilde{t}\right)\right) e^{-\alpha \tilde{t}}$, in view of the fact that $y(t)$ is a non-decreasing function of t (this inequality is strict for all $v > y(t)$). In other words, an immediate purchase leads to the largest surplus. Obviously, for any $v < y(t)$, we have $(v - y(s)) e^{-\alpha s} < 0$ for any $s \in [t, 1]$; thus purchasing is not valuable at any time.

Proof of Proposition 8. Obviously, among all equilibria in which the Q units are sold in the main season, the one with $p_1 = p_1^l$ yields the maximal revenue of $p_1^l \cdot Q$. It is left to show that any other type of equilibrium with $p_1 \leq p_1^l$, yields a revenue that is not larger than $p_1^l \cdot Q$. To verify this, suppose that an equilibrium where $p_1 \leq p_2$, and in which inventory is left at the end of the season, existed. This would mean that the customers would follow a policy $\theta(t) = p_1 e^{\alpha t}$, and that at the end of the season, the valuations of the remaining customers would be spread in the range $[0, p_1]$. But as a consequence, it would be optimal for the seller to set $p_2 < p_1$; a contradiction. Thus, any possible equilibrium with a premium price p_1 yields a revenue that is strictly lower than $p_1 \cdot Q$, and in particular, a revenue that is strictly smaller than $p_1^l \cdot Q$ for all $p_1 \in \left[0, p_1^l\right]$.

Proof of Proposition 9. First note that any $p_1 \geq p_1^u$ can lead to an equilibrium in which no customer purchases in the main season (i.e., $\theta = 1$) and where $p_2 = p_2^u \doteq e^{-\alpha} \max\left(\frac{1}{2}, 1 - \frac{Q}{\lambda}\right)$ – the best response of the seller to the strategy $\theta = 1$. To see that indeed $\theta = 1$ is optimal for such p_1, simply observe that at any time t, any customer with base valuation $v > p_1 e^{\alpha t}$ can gain the following additional surplus by waiting: $(ve^{-\alpha} - p_2^u) - (ve^{-\alpha t} - p_1) = (ve^{-\alpha} - ve^{-\alpha t}) - (p_2^u - p_1) \geq p_1 e^{\alpha t}(e^{-\alpha} - e^{-\alpha t}) - (p_2^u - p_1^u) = -p_1\left(1 - e^{-\alpha(1-t)}\right) + (1 - e^{-\alpha}) \geq (1 - p_1)(1 - e^{-\alpha}) \geq 0$. Next, we verify that no other equilibrium can exist when $p_1 \geq p_1^u$. To see this, suppose that we had an equilibrium with $p_1 \geq p_1^u$ and an end-of-season price p_2, where some customers would buy in the main season. Then, given the previous argument, this would mean that p_2 would have to be larger than p_2^u. In addition, since $p_2^u e^{\alpha} \leq p_1^u \leq p_1 \leq \theta(t) e^{-\alpha t}$, it is easy to see that $\int_0^1 (1 - \min(ue^{\alpha}, \theta(t)) \, dt \leq \int_0^1 (1 - p_2^u e^{\alpha}) \, dt \leq Q$ for *any* $u \geq p_2^u$. Now, let us consider the seller's optimization problem at the end of the season:

$$\max_{p_2} \left\{ p_2 \cdot \int_0^1 (\theta(t) - p_2 e^{\alpha})^+ \, dt \right\}$$
$$\text{s.t.,} \int_0^1 (1 - \min(p_2 e^{\alpha}, \theta(t)) \, dt \leq Q$$

But using straightforward calculus arguments, the value of p_2 that maximizes the above expression must be lower than p_2^u, the maximizer of the same optimization problem with $\theta(t)$ being substituted by 1. Hence a contradiction to the fact that p_2 must be larger than p_2^u.

Proof of Theorem 3. The proof of this theorem is obviously simple, given that the two-price strategy can be considered as a reward program (albeit not a surplus matching reward program) that offers null (zero) reward. Hence, it cannot achieve better performance than that of the optimal surplus-matching reward program.

Proof of Proposition 11. We first show that $p_1^{PM} = p_2^{PM}$ in equilibrium. Consider two possibilities: (i) obviously, if $p_2^{PM} \geq p_1^{PM}$, the consumers follow the strategy $\theta(t) = \min\left\{p_1^{PM}e^{\alpha t}, 1\right\}$, and no one would purchase at the end of the season. In such case, p_2^{PM} could simply be reset to p_1^{PM} without any effect on the consumers or the seller; (ii) if $p_2^{PM} < p_1^{PM}$, it must mean that some units are left for the end of the season, and p_2^{PM} does not induce a demand that is larger than the leftover inventory. Therefore, consumers do not face shortage risk, and it is always optimal for them to enjoy the product earlier (higher valuation) than later – given that they eventually pay the price p_2^{PM} (i.e., the consumers follows a strategy $\theta(t) = \min\left\{p_2^{PM}e^{\alpha t}, 1\right\}$). Furthermore, no consumer buys at the end of the season, since $\theta(t)e^{-\alpha} \leq p_2^{PM}$ for all t. Therefore, p_2^{PM} cannot be optimal in the subgame, as it merely offers a positive price match refund without the ability to generate further revenue. We hence conclude that in equilibrium, $p_1^{PM} = p_2^{PM}$. Next, we identify the optimal level of p_1^{PM}. For simplicity of notation, we will use p and p^* to denote the premium price and its optimal value, respectively. In the analysis below, keep in mind that the expression $1-\frac{1}{2}\alpha-e^{-\alpha}$ is positive for all $\alpha \in (0, \bar{\alpha})$ and negative for all $\alpha \in (\bar{\alpha}, \infty)$. Let us now consider the demand realization as a function of p:

$$D = \begin{cases} \lambda\int_0^1 \left(1-pe^{\alpha t}\right)dt = \lambda\left(1-p\frac{e^{\alpha}-1}{\alpha}\right) & p \leq e^{-\alpha} \\ \lambda\int_0^{-\frac{1}{\alpha}\ln(p)} \left(1-pe^{\alpha t}\right)dt = \frac{\lambda}{\alpha}\left(p-\ln(p)-1\right) & p > e^{-\alpha} \end{cases}$$

We proceed by identifying the solution to the unconstrained optimization (i.e., when inventory constraint is relaxed): When $p \leq e^{-\alpha}$, we get a concave (quadratic) function, so the best price within this range is $\frac{\alpha}{2}/\left(e^{\alpha}-1\right)$ if $\alpha \leq \bar{\alpha}$, and $e^{-\alpha}$ if $\alpha \geq \bar{\alpha}$. When $p \geq e^{-\alpha}$, the revenue function is given by $\frac{\lambda}{\alpha}\left(p^2-p\ln(p)-p\right)$, for which the derivative is given by $-2\frac{\lambda}{\alpha}\cdot\left(1+\frac{1}{2}\ln(p)-p\right)$. Using the above notation, the latter derivative is positive for all $p \in (0, e^{-\bar{\alpha}})$ and negative for all $p \in (e^{-\bar{\alpha}}, 1)$. Therefore, the best price within the range $p \geq e^{-\alpha}$ is given by $e^{-\alpha}$ if $\alpha \leq \bar{\alpha}$, and $e^{-\bar{\alpha}}$ if $\alpha \geq \bar{\alpha}$. Let us summarize: (i) for $\alpha \leq \bar{\alpha}$, we need to compare between $\frac{\lambda}{4}\frac{\alpha}{e^{\alpha}-1}$ and $\frac{\lambda}{\alpha}\left(e^{-\alpha}+\alpha-1\right)e^{-\alpha}$, and it is easy to verify that the former term dominates; i.e., $p^* = \frac{\alpha}{2}/\left(e^{\alpha}-1\right)$ if $\alpha \leq \bar{\alpha}$. The demand in this case equals to $\frac{\lambda}{2}$, and therefore the solution is feasible and optimal when $Q \geq \frac{\lambda}{2}$. (ii) When $\alpha \geq \bar{\alpha}$, we need to compare between $\lambda e^{-\alpha}\left(1-\frac{1-e^{-\alpha}}{\alpha}\right)$ and $\frac{\lambda}{\alpha}e^{-\bar{\alpha}}\left(e^{-\bar{\alpha}}-\ln\left(e^{-\bar{\alpha}}\right)-1\right)=\frac{\lambda}{\alpha}e^{-\bar{\alpha}}\frac{\bar{\alpha}}{2}$. Utilizing simple algebra, we can show that the latter expression dominates; i.e., $p^{*}=e^{-\bar{\alpha}}$ if $\alpha \geq \bar{\alpha}$. The solution for this case is feasible when $Q \geq \frac{\lambda}{2}\cdot\frac{\bar{\alpha}}{\alpha}$. The rest of the proposition follows trivially, given that the revenue functions and the quantity sold continuously decline as we increase p above its unconstrained optimal level.

Proof of Proposition 12. Follows immediately from the proof of Proposition 11, by plugging in the optimal prices into the revenue expression.

Proof of Proposition 13. The proof that the consumers' response to the announced price path (p_1, p_2) and a perceived value of β is $\theta(t)=\max\left(\frac{p_1-\beta\cdot p_2}{e^{-\alpha t}-\beta\cdot e^{-\alpha}}, p_1 e^{\alpha t}\right)$ follows simple algebra in a way parralel to the proof of Proposition 10. Taking the derivative of the first term in θ with respect to β, we get: $\partial\left(\frac{p_1-\beta\cdot p_2}{e^{-\alpha t}-\beta\cdot e^{-\alpha}}\right)/\partial\beta=\frac{1}{\left(e^{-t\alpha}-\beta e^{-\alpha}\right)^2}\left(p_1 e^{-\alpha}-p_2 e^{-t\alpha}\right)$, which is non-negative whenever θ is determined by this term. This verifies that $\theta(t)$ is non-decreasing in β for all t. Next, it is easy to verify that the total demand induced by prices (p_1, p_2) is given by $X \doteq \lambda\int_0^1\left(1-\min\left(p_1 e^{\alpha t}, p_2 e^{\alpha}, 1\right)\right)^{+} dt$, and consequently the influence of θ on β is takes place by how the demand is split in between the main season and the end of the season, but not via the total demand X. Thus, it is straightforward to see that β is a strictly increasing function of θ. More specifically, let $\theta_0(t)=p_1 e^{\alpha t}$, and calculate the value of β_0 using

$$\beta=\min\left(\frac{\frac{Q}{\lambda}-\int_0^1(1-\theta(t))^{+} dt}{\int_0^1\left(\theta(t)-p_2 e^{\alpha}\right)^{+} dt}, 1\right) \tag{12}$$

Then, iteratively continue by setting $\theta_i(t)=\max\left(\frac{p_1-\beta_{i-1}\cdot p_2}{e^{-\alpha t}-\beta_{i-1}\cdot e^{-\alpha}}, p_1 e^{\alpha t}\right)$ and calculate β_i using the (12). Since $\beta_0 \leq \beta_1 \leq \ldots \leq \beta_i \leq \ldots \leq 1$, convergence to a unique value of β is guaranteed.