On the Worst-Case Performance of Uniform Co-payments in Market Consumption

Gonzalo Romero, Retsef Levi, Georgia Perakis

On the Worst-Case Performance of Uniform Co-payments in Maximizing Market Consumption

(Authors' names blinded for peer review)

We consider a subsidy allocation problem with endogenous market response and a budget constraint. The central planner's objective is to maximize the market consumption of a good, and she allocates per-unit co-payments to its producers. We provide the *first worst-case performance guarantees for uniform co-payments* in this setting. That is, the simple policy that allocates the same co-payment to every firm is guaranteed to induce a significant fraction of the market consumption induced by optimal co-payments, even if the firms are heterogeneous and their efficiency levels are arbitrarily different. This is an important insight because uniform co-payments is the policy most commonly implemented in practical applications of this problem. Moreover, this is even more relevant as we show that the optimal co-payments structure is impractical. Specifically, for *any* decreasing inverse demand function, uniform co-payments induce *at least half* of the optimal market consumption, for price taking producers with affine increasing marginal costs. Furthermore, for Cournot competition with linear demand and constant marginal costs, this guarantee increases to more than 85% of the optimal market consumption. Our results suggest that uniform co-payments are a surprisingly powerful policy to increase the market consumption of a good.

Key words: Subsidies, Worst-case analysis, Market competition

1. Introduction

In many relevant settings, the aggregated market consumption of a good, induced by the competition between selfish producers, is less than what is considered socially optimal. This is generally due to the positive societal externalities generated by its consumption, which, by definition, are not internalized by consumers. Classical examples of such goods include vaccines and infectious disease treatments, see Brito et al. (1991) and Arrow et al. (2004), respectively.

One frequently implemented method to address this problem, is having a central planner intervene the market by allocating subsidies to the producers of the good, with the objective of increasing its market consumption. A current example are the recent efforts around the production of infectious disease treatments to the developing world, such as antimalarial drugs (e.g., Arrow et al. (2004)), and vaccines (e.g., Snyder et al. (2011)). In most of these cases, the type of subsidy being used are fixed per-unit subsidies, or *co-payments*. Furthermore, typically the central planer allocates the co-payments in the presence of a budget constraint, which is often determined before the allocation decision is made. For example, the role of the central planner could be played by a foundation that has raised a given amount of money in order to address the low consumption of an infectious disease treatment, and then faces the problem of deciding how to allocate this budget as co-payments. An additional challenge faced by the central planner is that her co-payments allocation will likely change the market equilibrium attained by the competing producers. Therefore, in order to maximize the market consumption induced by her co-payments allocation, the central planer has to take into account these potentially complex market dynamics.

The co-payment allocation policy most often implemented in practice is *uniform*, in the sense that every firm gets the same co-payment, regardless of any differences in their cost structure or efficiency. This is probably due to the simplicity, transparency, and ease of implementation of this policy. How close the market consumption induced by uniform co-payments is to the one induced by optimal co-payments can be the key to effectively correct market imperfections, such as the ones observed in markets with large positive externalities. This issue is even more relevant if one assumes that some firms may be much more efficient than others, motivating the goal of understanding the optimal co-payment allocation structure, and providing insights on how suboptimal uniform co-payments can be in the worst-case.

A current practical example of the setting previously described is the global fight against malaria. It is estimated that in 2012 about 200 million cases occurred worldwide, and more than 600,000 people died of malaria, see the world malaria report by the World Health Organization (2013). Recently, a treatment called Artemisinin combination therapies (ACT) has been identified as the presently most effective treatment for malaria; however, its market price in African countries is at least ten times larger than less effective alternatives, see White (2008). For this reason, by July

2012 UNITAID, the governments of the United Kingdom and Canada, and the Bill & Melinda Gates Foundation, pledged a budget of US$336 millions, which has been managed by the Global Fund with the goal of increasing the consumption of ACTs, as detailed in the evaluation report by the AMFm Independent Evaluation Team (2012). As it is frequently the case in practice, the policy implemented was to allocate this budget in the form of a *uniform co-payment* to the producers of ACTs. Specifically, each of the 11 producers participating in the program receives the same co-payment for each unit, regardless of their individual characteristics. The list of producers is highly heterogeneous, both in their market size and location-wise. It ranges from large pharmaceuticals like Novartis (having manufacturing plants in USA and China) and Sanofi (having manufacturing plants in Germany and Morocco), to smaller firms with manufacturing plants in Uganda, India and Korea (for more details see the market intelligence aggregator, funded by UNITAID, A2S2 (2014)). An additional characteristic of this setting is that all the ACT producers that receive co-payments commit to supply anti-malarials on a no profit/no loss basis, see the report by Boulton (2011). Finally, note that the Global Fund is no position of charging taxes to the producers, as they operate in different countries. The Global Fund can only allocate incentives, and it has decided to do so by implementing uniform co-payments.

In this paper, we present the *first worst-case guarantees* for the performance of uniform co-payments in maximizing the market consumption of a good. This provides theoretical support for both continuing with the implementation of this policy in practical applications, as well as giving a plausible explanation for the good performance results observed in simulations in previous papers, see The authors (2014). Specifically, we consider a model that explicitly captures the setting of a central planner aiming at maximizing the market consumption of a good, in the presence of a budget constraint and market competition between heterogeneous profit maximizing firms. The firms are heterogeneous in terms of their respective marginal cost functions, which model their firm-specific efficiency.

In particular, the models studied in this paper capture general decreasing inverse demand functions, for price taking producers with arbitrarily different affine increasing marginal cost functions. The generality of an arbitrary decreasing inverse demand function allows to model complex downstream demand mechanisms that have been considered in the operations management literature, including the case of multiple competing retailers under demand uncertainty. Additionally, this class of models includes, as a special case, important settings with imperfect competition dynamics, such as Cournot competition with linear demand and affine marginal costs. For this fairly general model, the paper provides surprising insights. First, we can show that uniform co-payments are guaranteed to induce *at least half* of the market consumption induced by optimal co-payments.

This is an important result, as we additionally show that the optimal co-payments have a complicated structure that makes them impractical. Furthermore, for the well studied special case of Cournot competition with linear demand and constant marginal costs, we derive a surprisingly high worst-case performance guarantee of more than 85% of the optimal market consumption. Additionally, we also present insights into necessary conditions for uniform co-payments to have a bounded performance, as well as sufficient conditions that extend their bounded performance to more general settings, where the firms may have non-linear increasing marginal costs functions. Our results suggest that the efficiency loss induced by uniform co-payments in maximizing the market consumption of a good can be expected to be relatively small in a large number of settings. Hence, this bounded efficiency loss should be weighted against the important practical advantages of uniform co-payments, such as their transparency and ease of implementation and communication, supporting the use if this policy in practice.

1.1. Results and Contributions

The main contributions of this paper are the following:

Insights into the structure of optimal co-payments. We characterize the optimal co-payment allocation, for price taking producers with affine increasing marginal costs facing *any* decreasing inverse demand function. We show that it consists of giving *larger co-payments to less efficient firms*. Moreover, we also show that the values of the optimal co-payments are complicated functions of the marginal cost parameters of each firm, as well as being very sensitive to misspecifications of these values. We argue that this policy is hard to implement, further supporting the use of the more practical, and frequently implemented, uniform co-payments.

Worst-case performance guarantee for a general model. For the previously described model, we show that uniform co-payments are guaranteed to induce *at least half* of the market consumption induced by optimal co-payments. Moreover, we show that this worst-case bound is *asymptotically tight* and cannot be improved. This result is surprising, as it holds for *any* decreasing inverse demand function, and for price taking producers having affine increasing marginal costs with arbitrarily different levels of efficiency. Furthermore, we derive sufficient conditions that extend this worst-case performance guarantee to more general settings, where the firms may have non-linear increasing marginal costs functions. These results suggest that the very simple uniform co-payments policy is likely to provide most of the benefits, in terms of maximizing the market consumption of a good, of more sophisticated firm-dependent co-payment allocation policies.

Detailed analysis for a relevant special case. For the important case of Cournot competition with linear demand and constant marginal costs, we show a surprisingly high 85.36% worst-case performance guarantee for uniform co-payments in maximizing the market consumption of a good. Namely, we show that *for each instance* of this well understood oligopoly model, uniform co-payments will always induce *at least* 85.36% of the market consumption induced by optimal co-payments. Again, we show that this worst-case bound is *asymptotically tight* and cannot be improved. Furthermore, for any fixed number of firms in the market, we derive *closed form* expressions for both the worst-case instance, and for the relative performance of uniform co-payments, obtaining even higher guarantees.

Necessary conditions for uniform co-payments to have a bounded performance. We additionally consider a model where the central planner is allowed to charge per-unit taxes to the producers, as well as co-payments. We show that the worst-case performance of uniform co-payments in maximizing market consumption is unbounded in this model, even for the simple case of price taking firms with affine marginal costs with the same slope facing linear demand.

In summary, our results suggest that in many practical settings, like the case of ACT anti-malarials discussed earlier, where the central planner allocates co-payments to the producers and she is in no position of charging taxes, then *uniform co-payments are a surprisingly powerful policy* to increase the market consumption of a good.

2. Literature Review

In economics, there exists a large literature that studies the effect of taxes and subsidies. Fullerton and Metcalf (2002) present a thorough review of classical and recent result in this area. This paper is closely related to the study of subsidies in imperfect competition models. However, the traditional approach in this literature assumes homogeneous firms, and focuses on studying the impact of taxes, or subsidies, on the number of firms participating in the market in a symmetric equilibrium, see Fullerton and Metcalf (2002). In contrast, the model in this paper has an operational approach, in the sense the producers are assumed to be heterogeneous, and we focus on the specific subsidy allocation among them, and its dependence on the available budget.

The co-payments allocation problem we consider in this paper was introduced in The authors (2014), motivated by the practical case of the global subsidy for ACT anti-malarial drugs. Using this framework, sufficient conditions on the firms' marginal cost functions were identified such that uniform co-payments are optimal. Additionally, simulation results in relevant settings where uniform co-payments are not optimal suggested that they are nonetheless very effective. This paper provides theoretical support for these observations, by showing the first worst-case performance guarantees for uniform co-payments in maximizing the market consumption of a good.

As already mentioned in the Introduction, the fact that our model supports a general decreasing inverse demand function allows the modeling of complex demand mechanisms from the operations management literature. One such example is the case of multiple competing retailers under demand uncertainty. Specifically, Bernstein and Federgruen (2005) have shown that in a model where each retailer chooses its retail price and its order quantity, and faces multiplicative random demand -the distribution of which may depend on its own retail price as well as those of the other retailers- there exists a unique Nash equilibrium, in which all the retailer prices decrease when the wholesale price is reduced. Moreover, under additional mild assumptions this leads to each equilibrium order quantity being decreasing in the wholesale price, resulting in a decreasing inverse demand function, and thus matching our modeling approach.

The problem of allocating subsidies to increase the market consumption of ACT anti-malarial drugs was studied independently by Taylor and Xiao (2014). They consider the case of one producer selling to multiple heterogeneous retailers facing stochastic demand, and analyze the placement of the subsidy by the central planner in the supply chain. Specifically, they compare subsidizing sales or purchases from the retailers' point of view, concluding that the central planner should only subsidy purchases, which is equivalent to subsidizing the producer in our setting. We consider a different model where we incorporate multiple heterogeneous producers. Our results suggest that not only allocating co-payments to the producers makes sense as an strategy to maximize the market consumption of a good, but the very simple, and frequently implemented, uniform co-payments are surprisingly effective, even if the producers are highly heterogeneous. A different, but related, area of research in operations management studies the problem of a central planner deciding rebates that are directed to the *consumers*, with the goal of incentivizing the adoption of green technology (see Aydin and Porteus (2009), Lobel and Perakis (2012), Cohen et al. (2012), and Krass et al. (2013)). In contrast, our work is motivated by a different set of practical applications, and it focuses on co-payments that are allocated to the producers.

The special case of Cournot competition with linear demand and constant marginal costs is a simple oligopoly model where the firms compete in quantity. It is a well understood model that provides interesting insights. Therefore, it is frequently used by researchers as a building block to study complex phenomena. Examples of this trend in the operations management, and operations research, literature include using this model, among others, to study the structure of supply chains, see Corbett and Karmarkar (2001), supply chain contracts, see Cachon (2003), production under yield uncertainty, see Deo and Corbett (2009), firms' profits compared to other equilibrium concepts, see Farahat and Perakis (2011), and facility network design under competition, see Dong et al. (2013). We provide a detailed analysis of the worst-case performance of uniform co-payments in maximizing the market consumption of a good, for this important model.

3. Model

In this section, we describe a mathematical programming formulation for the problem of allocating co-payments to competing firms producing a commodity. We consider a market for a commodity composed by $n \geq 2$ heterogeneous competing firms. We assume that each firm $i \in \{1, \ldots, n\}$ decides its output $q_i \geq 0$ independently, with the goal of maximizing its own profit, and that it has an affine increasing marginal cost $h_i(q_i) = c_i + g_i q_i$, where $c_i \geq 0$ and $g_i > 0$. Consumers are described by an inverse demand function $P(Q)$, where $Q \equiv \sum_{i=1}^n q_i$ is the aggregated market consumption. We assume that $P(Q)$ is non-negative, decreasing and differentiable in $[0, \bar{Q}]$, where $\bar{Q}$ is the smallest value such that $P(\bar{Q}) = 0$. We will also assume, without loss of generality, that the firms are labeled such that $c_1 \leq c_2 \leq \ldots \leq c_n < P(0)$.

In terms of the market equilibrium dynamics, we assume that each firm participating in the market equilibrium produces up to the point where its marginal cost equals the market price; and firms that do not participate in the market equilibrium must have a marginal cost of producing zero units which is larger than the market price. This can be expressed in the following condition:

$$\text{For each } i, j, \text{ if } \; q_i > 0, \text{ then } \; c_i + g_i q_i = P(Q) \leq c_j + g_j q_j. \tag{1}$$

The assumption that the inverse demand function $P(Q)$ is decreasing, and that the firms' marginal cost functions are increasing, implies the existence and uniqueness of the market equilibrium, see for example Marcotte and Patriksson (2006).

Some special cases of the market equilibrium condition (1) include imperfect market competition models, such as Cournot Competition with linear demand and affine marginal costs, even if we incorporate yield uncertainty, we refer the reader to The authors (2014) for the details. Another special case of condition (1) is the model where the firms act as price takers and compete in quantity, for any decreasing inverse demand function. In general, the model where the firms act as price takers is a reasonable approximation whenever the firms in the market have little market power, for example when there are many firms competing in the market, or when firms face the threat of entry to their market, see Tirole (1988).

As already mentioned in Section 1, we focus on settings where the market consumption induced at the market equilibrium is less than what is socially optimal. For this reason, a central planner intervenes the market by allocating a fixed per-unit subsidy, or co-payment, to each firm. We will refer to the problem faced by the central planner as the *co-payment allocation problem (CAP).* The co-payment allocation problem is a particular case of a Stackelberg game, see Stackelberg (1952), or a bilevel optimization problem. In the first stage, the central planner allocates her budget $B > 0$, in the form of co-payments $y_i \geq 0$ per unit provided in the market, to each firm $i \in \{1, \ldots, n\}$. Moreover, she anticipates that in the second stage the equilibrium output of each firm will satisfy a modified version of the equilibrium condition (1), stated below in constraint (6). The main difference in the market equilibrium condition (6), with respect to (1), is given by the fact that, from firm i's perspective, the effective price for each unit sold is now $P(Q) + y_i$.

The central planner's objective is to maximize the aggregated market consumption. Finally, note that we have assumed that the central planner can only allocate co-payments, and never charge taxes for the units produced in the market. In other words, the co-payments must be non-negative. As already mentioned, this is the case in most practical instances of this problem. Specifically, the central planner almost never has the authority to charge taxes to firms that operate in different countries, with the goal of increasing the aggregated market consumption of a good, see for example Arrow et al. (2004) for the case of anti-malarials. It follows that the co-payments allocation problem faced by the central planner can be formulated as

$$\begin{aligned}
\max_{\mathbf{y},\mathbf{q},Q} \quad & Q \\
\text{s.}t. \quad & \sum_{i=1}^{n} q_i y_i \le B && (2)\\
& y_i \ge 0, \text{ for each } i \in \{1,\dots,n\} && (3)\\
& \sum_{i=1}^{n} q_i = Q && (4)\\
& q_i \ge 0, \text{ for each } i \in \{1,\dots,n\} && (5)\\
& c_i + g_i q_i = P(Q) + y_i, \text{ for each } i \in \{1,\dots,n\}. && (6)
\end{aligned}$$

Constraint (2) is the budget constraint, where the total amount spent in co-payments must be at most the available budget B. Constraints (3) and (6) are the non-negativity of the co-payments, and the modified equilibrium condition previously discussed, respectively.

Note that this is a valid formulation even if there are firms that do not participate in the market equilibrium. Namely, if for some firm i we have $q_i = 0$, then, from the modified market equilibrium condition (6), and the non-negativity of the co-payments (3), we must have $y_i = c_i - P(Q) \ge 0$. Hence, the non-negativity of the co-payments constraint (3), exactly ensures that the original market equilibrium condition (1) is satisfied. Moreover, allocating a co-payment $y_i = c_i - P(Q) \ge 0$ to firm i is without loss of generality, because setting $q_i = 0$ ensures that firm i does not have an impact in the budget constraint (2). In other words, the fact that we impose the modified market equilibrium condition (6) *on each* firm $i \in \{1,\dots,n\}$, does *not* imply that every firm must join the market equilibrium.

From the equilibrium condition (6), it follows that we can replace all the co-payment variables y_i by $c_i + g_i q_i - P(Q)$. In other words, we can reformulate the co-payment allocation problem as if the central planner was deciding the output of each firm, as long as there exist feasible co-payments that can sustain the outputs chosen as a market equilibrium. The feasibility of the co-payments is given by both the budget constraint (2), and the non-negativity of the co-payments (3). Hence, we conclude that the following is a valid reformulation of the co-payment allocation problem faced by the central planner

$$
(CAP) \qquad
\begin{aligned}
\max_{\mathbf{q},Q} \quad & Q \\
\text{s.}t. \quad & \sum_{i=1}^{n}(c_i q_i + g_i q_i^2) - P(Q)Q \le B && (7)\\
& c_i + g_i q_i \ge P(Q), \text{ for each } i \in \{1,\dots,n\} && (8)\\
& \sum_{j=1}^{n} q_j = Q && (9)\\
& q_i \ge 0, \text{ for each } i \in \{1,\dots,n\}, && (10)
\end{aligned}
$$

where constraint (7) is equivalent to the budget constraint (2), and constraint (8) is equivalent to the non-negativity of the co-payments (3). The co-payments that the central planner must allocate, in order to induce the selected outputs $\mathbf{q}$, are $y_i = c_i + g_i q_i - P(Q)$, for each firm $i \in \{1,\dots,n\}$.

In the remainder of this section, we show that we can characterize the market consumption induced by optimal co-payments and uniform co-payments, respectively.

3.1. Optimal Co-payments

In this section, we characterize the structure of the market equilibrium induced by uniform co-payments. Specifically, Proposition 1 below shows that the market outputs induced by the optimal co-payments have the following intuitive structure: more efficient firms produce more than less efficient firms, in the sense that if one active firm i dominates an active firm j in both of the marginal cost function's parameters, namely $c_i \le c_j$ and $g_i \le g_j$, then firm i will have a larger market output. This holds up to the point where the firms are so inefficient, in terms of the value of their parameter c_i, that they do not participate in the market equilibrium induced by optimal co-payments. The latter is characterized by an index $m \in \{1,\dots,n\}$, associated to the last firm that has a positive output in this market equilibrium.

Similarly, the optimal co-payments have the following structure. The more efficient firms, in terms of having a smaller value of its parameter c_i, may get no co-payments, and only after some index $l \in \{1,\dots,m\}$, firms start getting a co-payment that is increasing in their marginal cost parameter c_i (see Proposition 1 below). Namely, in order to maximize the market consumption at equilibrium, for price taking firms with affine increasing marginal costs, facing an arbitrary decreasing inverse demand function, the best that the central planner can do with her co-payments is to give *more co-payments to less efficient firms*. This structural result is driven by the central planner's objective of maximizing the market consumption. For example, it can be shown that if the central planner's objective was to minimize the total cost instead, then we would obtain the opposite result, where more efficient firms would get a larger co-payment at optimality.

Proposition 1. *Any optimal solution of the co-payments allocation problem (CAP), $(\boldsymbol{q}^*, Q^*)$, must be such that the budget constraint (7) is tight, and there exist indexes $l, m \in \{1, \ldots, n\}$, with $l \leq m$, such that the optimal co-payments are given by*

$$y_i^* = 0, \text{ for each } i \in \{1, \ldots, l-1\}, \tag{11}$$

$$y_i^* = y_l^* + \frac{c_i - c_l}{2} > 0, \text{ for each } i \in \{l, \ldots, m\}, \tag{12}$$

$$y_i^* = y_l^* + c_i - c_l - g_l q_l^* \geq y_l^* + \frac{c_i - c_l}{2} > 0, \text{ for each } i \in \{m+1, \ldots, n\}. \tag{13}$$

The optimal market outputs are given by

$$q_i^* = \frac{g_l}{g_i} q_l^* + \frac{c_l - c_i}{g_i} - \frac{y_l^*}{g_i} \geq \frac{g_l}{g_i} q_l^* + \frac{c_l - c_i}{2g_i} > 0, \text{ for each } i \in \{1, \ldots, l-1\}, \tag{14}$$

$$q_i^* = \frac{g_l}{g_i} q_l^* - \frac{c_i - c_l}{2g_i} > 0, \text{ for each } i \in \{l, \ldots, m\}, \tag{15}$$

$$q_i^* = 0, \text{ for each } i \in \{m+1, \ldots, n\}, \tag{16}$$

$$q_l^* = \frac{P(Q^*) + y_l^* - c_l}{g_l}. \tag{17}$$

The expressions (12)-(17) are written as a function of the first positive co-payment y_l^*, which is such that

$$\sum_{i=l}^{m} \frac{y_l^*}{g_i} = \sqrt{\left(\sum_{i=l}^{m} \frac{P(Q^*) - c_i}{2g_i}\right)^2 + \sum_{i=l}^{m} \frac{B}{g_i} + \sum_{i=l}^{m} \frac{1}{g_i} \left(\sum_{i=l}^{m} \left(\frac{c_i - c_l}{2}\right)^2 \frac{1}{g_i}\right) - \left(\sum_{i=l}^{m} \frac{c_i - c_l}{2g_i}\right)^2} - \sum_{i=l}^{m} \frac{P(Q^*) - c_l}{2g_i}. \tag{18}$$

Finally, the aggregated market consumption must satisfy the following fixed point equation

$$Q^* = \sqrt{\left(\sum_{i=l}^{m} \frac{P(Q^*) - c_i}{2g_i}\right)^2 + \sum_{i=l}^{m} \frac{B}{g_i} + \sum_{i=l}^{m} \frac{1}{g_i} \left(\sum_{i=l}^{m} \left(\frac{c_i - c_l}{2}\right)^2 \frac{1}{g_i}\right) - \left(\sum_{i=l}^{m} \frac{c_i - c_l}{2g_i}\right)^2} + \sum_{i=1}^{m} \frac{P(Q^*) - c_i}{g_i} - \sum_{i=l}^{m} \frac{P(Q^*) - c_i}{2g_i}. \tag{19}$$

The proof of Proposition 1 follows from the KKT conditions of problem (CAP), which can be shown to be necessary for optimality, and is omitted.

Practical Challenges. Proposition 1 characterizes the optimal co-payment y_i^*, and the induced market output q_i^*, for each firm $i \in \{1, \ldots, n\}$, as well as giving a fixed point equation that must be satisfied by the induced market consumption Q^*. Moreover, it provides closed form expressions that are parametrized by the indexes l, of the first firm that receives a co-payment, and m, of the last firm that has a positive market output. Nonetheless, if we wanted to transfer these insights into practice, we would have to keep in mind that the optimal co-payments policy imposes the following challenges. First, the optimal co-payments are a complicated function of the problem parameters, which would make them difficult to communicate. Second, they are different for each firm, which would significantly increase the complexity of the process of paying to the producers. More importantly, the optimal co-payments policy requires the central planner to know the marginal cost functions $h_i(q_i) = c_i + g_i q_i$ of *each* firm i, as well as being highly sensitive to changes in the value of the marginal cost functions' parameters. Although in our model we assumed a full information setting, where the central planner knows the marginal cost function of each firm, in practice this may not be the case. Therefore, any practical implementation would either require a truthful mechanism to elicit the marginal cost functions, or alternatively it would have to deal with potential misspecifications. In contrast, the uniform co-payments policy is simple to communicate and control. Additionally, we will see in the next section that it only depends on the *average* of simple functions of the parameters of the marginal cost of the firms, hence it is more stable to potential misspecifications. These characteristics make the uniform co-payments a more attractive policy for practical purposes, as long as the loss in the induced market consumption, with respect to the optimal co-payments policy, is not very large.

3.2. Uniform Co-payments

In this section, we characterize the structure of the market equilibrium induced by uniform co-payments. By definition, the uniform co-payments policy allocates the same co-payment to each firm. Naturally, we focus on the largest possible uniform co-payment that can be afforded with the central planner's budget B. Specifically, if we denote by q_i^{U} firm i's output induced by the uniform co-payment y^{U}, then we will focus on the value of y^{U} such that $\sum_{i=1}^n q_i^{\mathrm{U}} y^{\mathrm{U}} = B$, or equivalently $y^{\mathrm{U}} = \frac{B}{Q^{\mathrm{U}}}$.

Namely, the amount of the uniform co-payment is obtained simply by dividing the available budget B, by the largest market consumption that be can attained with this budget under a uniform co-payment policy, denoted by Q^{U}. In practice, the way this policy is usually implemented is by dividing the budget by a target market consumption that the central planner has set as a goal, see for example AMFm Independent Evaluation Team (2012) for the case of ACT anti-malarials. In terms of the parameters in our model, the structure of the market equilibrium induced by uniform co-payments is described in the following lemma.

Q^{U}/Q^*	n=2	n=3	n=10	n=20
Min.	0.9360	0.9175	0.9182	0.9442
1st Qu.	0.9776	0.9734	0.9785	0.9828
Median	0.9919	0.9845	0.9849	0.9876
Mean	0.9860	0.9806	0.9834	0.9866
3rd Qu.	0.9983	0.9930	0.9902	0.9912
Max.	1.0000	1.0000	1.0000	0.9991

Table 1 Relative Performance of Uniform Co-payments - Cournot Constant MC

LEMMA 1. *Assume that the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0,\bar{Q}]$, and that the affine marginal cost function of each firm $i \in \{1,\ldots,n\}$ is $h_i(q_i) = c_i + g_i q_i$, where $c_i \geq 0$ and $g_i > 0$. Then, the market consumption induced by the uniform co-payments allocation of the budget $B \geq 0$, Q^{U}, is the unique solution to the following fixed point equation*

$$Q^{\mathrm{U}} = \sum_{i=1}^{u} \frac{P(Q^{\mathrm{U}}) - c_i}{2g_i} + \sqrt{\left(\sum_{i=1}^{u} \frac{P(Q^{\mathrm{U}}) - c_i}{2g_i}\right)^2 + \sum_{i=1}^{u} \frac{B}{g_i}}, \tag{20}$$

where the index $u \in \{1,\ldots,n\}$ is such that $\sum_{i=1}^{u} \frac{c_u - c_i}{g_i} \leq Q^{\mathrm{U}} \leq \sum_{i=1}^{u} \frac{c_{u+1} - c_i}{g_i}$. where we define, without loss of generality, $c_{n+1} = P(0)$. Similarly, the firms' outputs q_i^{U} are

$$q_i^{\mathrm{U}} = \frac{Q^{\mathrm{U}} + \sum_{j=1}^{u} \frac{c_j}{g_j}}{g_i \sum_{j=1}^{u} \frac{1}{g_j}} - \frac{c_i}{gi} \geq 0 \quad \text{for each } i \in \{1,\ldots,u\}, \quad q_i^{\mathrm{U}} = 0, \ \text{for each } i \in \{u+1,\ldots,n\}. \tag{21}$$

The uniform co-payment that induces this market output is $y_i^{\mathrm{U}} = \dfrac{B}{Q^{\mathrm{U}}} \geq 0$, for each $i \in \{1,\ldots,u\}$.

The proof of Lemma 1 follows directly from the market equilibrium condition (6), and the fact that each active firm receives the same uniform co-payment $y_i^{\mathrm{U}} = \frac{B}{Q^{\mathrm{U}}}$, and is omitted.

Note that the fixed point equation for the market consumption induced by uniform co-payments Q^{U} in equation (20) depends only on averages of simple functions of the marginal cost functions' parameters of the firms that are active in the market equilibrium. Specifically, it only depends on $\sum_{i=1}^{u} \frac{1}{g_i}$ and $\sum_{i=1}^{u} \frac{c_i}{g_i}$. This makes the value of Q^{U} more robust to changes or misspecifications of the values of the marginal cost parameters, when compared to the market consumption induced by the optimal co-payments, Q^*.

3.3. Numerical Results

In this section, we motivate the use of uniform co-payments in maximizing market consumption by numerically studying their performance. In particular, we consider the important special case of Cournot competition with linear demand and constant marginal costs. The market outputs in this market equilibrium are equivalent to the ones attained by price taking firms with marginal cost functions $h_i(q_i) = c_i + bq_i$, facing a linear inverse demand function $P(Q) = a - bQ$, see The

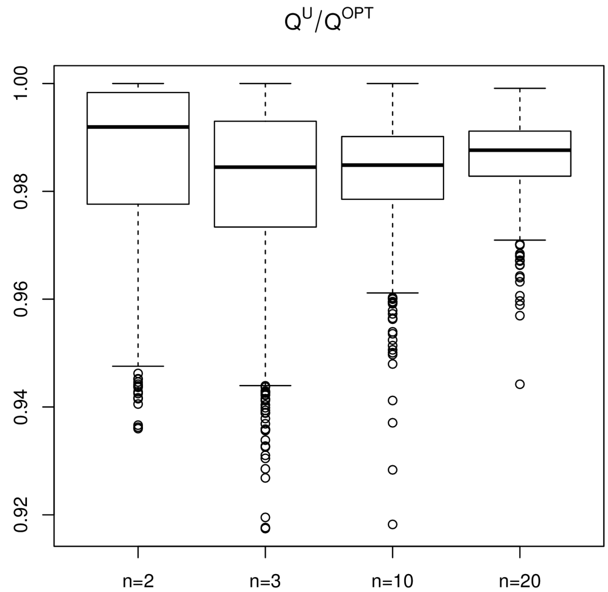

Figure 1 **Relative Performance of Uniform Co-payments - Cournot Constant MC**

authors (2014). Under these assumptions, the co-payment allocation problem (CAP) is a convex optimization problem. Therefore, we can compare the market consumption induced by optimal co-payments Q^* to the market consumption induced by uniform co-payments Q^{U}.

As in The authors (2014), we consider four cases in the number of firms participating in the market, $n \in \{2, 3, 10, 20\}$. We solve 1,000 instances for each one of these cases. These instances are randomly generated, where a, b are uniformly distributed in $[0, 50]$, and each c_i is uniformly distributed in $[0, a]$.

Figure 1 presents a boxplot of the results for the ratio Q^{U}/Q^*, while Table 1 presents some summary statistics. It is interesting that the minimum value of the ratio Q^{U}/Q^* never went below 91% in the simulation results. Moreover, the mean and median values are above 98%, for each value of the number of firms participating in the market n. This suggests that, in most cases, the market consumption induced by uniform co-payments is fairly close to the market consumption induced by the optimal co-payments allocation.

4. Worst-case Performance Guarantee of One Half

In this section we show an *asymptotically tight* worst-case guarantee of one half for the performance of uniform co-payments in maximizing the market consumption of a good, for price taking producers with affine increasing marginal costs, facing an arbitrary decreasing inverse demand function. As already described, this class of models generalizes important imperfect competition models, such as Cournot competition with linear demand and affine marginal costs. We start with an observation that allows to simplify the analysis. All the proofs are presented in Appendix A.

Proposition 2. *Let $l, m, u \in \{1, \ldots, n\}$ be indexes as defined in Proposition 1 and Lemma 1, respectively. Then, for any given number of firms in the market $n \geq 2$, in order to study the worst-case performance of uniform co-payments there is no loss of generality in focusing on instances of the problem (CAP) such that $l \leq m = u = n$.*

Now we operationalize the result in Proposition 2. Namely, we would like to characterize the instances of problem (CAP) that satisfy the condition $l \leq m = u = n$, as there is no loss of generality in focusing on them in order to study the worst-case performance of uniform co-payments.

Lemma 2. *Let $l, m, u \in \{1, \ldots, n\}$ be indexes as defined in Proposition 1 and Lemma 1, respectively. Then, for any given number of firms in the market $n \geq 2$, an instance of problem (CAP) is such that $l \leq m = u = n$ if and only if it satisfies the following conditions*

$$y_l^* \geq 0 \Leftrightarrow B - \sum_{i=l}^{n} \left(\frac{2P(Q^*) - c_i - c_l}{2} \right) \left(\frac{c_i - c_l}{2g_i} \right) \geq 0. \tag{22}$$

$$y_l^* \leq \frac{c_l - c_{l-1}}{2} \Leftrightarrow \sum_{i=l}^{n} \left(\frac{2P(Q^*) - c_i - c_{l-1}}{2} \right) \left(\frac{c_i - c_{l-1}}{2g_i} \right) - B \geq 0. \tag{23}$$

$$q_n^* \geq 0 \Leftrightarrow B - \sum_{i=l}^{n} \left(\frac{c_n - c_i}{2} \right) \left(\frac{c_n + c_i - 2P(Q^*)}{2g_i} \right) \geq 0. \tag{24}$$

$$q_n^{\mathrm{U}} \geq 0 \Leftrightarrow B - \left(c_n - P(Q^{\mathrm{U}})\right) \left(\sum_{i=1}^{n} \frac{c_n - c_i}{g_i} \right) \geq 0. \tag{25}$$

The proof of Lemma 2 is omitted. It consists on verifying that conditions (22)-(23) are equivalent to imposing the non-negativity of the co-payments (8), and conditions (24) and (25) are equivalent to imposing the non-negativity of the market outputs (10).

From Proposition 2 and Lemma 2 it follows that in order to study the worst-case performance of uniform co-payments, without loss of generality we can focus on instances of problem (CAP) that satisfy conditions (22)-(25), as well as the assumptions of the model: $B \geq 0$ and $P(Q^{\mathrm{U}}) \geq P(Q^*)$. Our goal now is to show that any such instance must satisfy

$$2Q^{\mathrm{U}} - Q^* \geq 0 \Leftrightarrow -\sqrt{\left(\sum_{i=l}^{n} \frac{P(Q^*) - c_i}{2g_i} \right)^2 + \sum_{i=l}^{n} \frac{B}{g_i} + \sum_{i=l}^{n} \frac{1}{g_i} \left(\sum_{i=l}^{n} \left(\frac{c_i - c_l}{2} \right)^2 \frac{1}{g_i} \right) - \left(\sum_{i=l}^{n} \frac{c_i - c_l}{2g_i} \right)^2}$$
$$2\sqrt{\left(\sum_{i=1}^{n} \frac{P(Q^{\mathrm{U}}) - c_i}{2g_i} \right)^2 + \sum_{i=1}^{n} \frac{B}{g_i}} + \sum_{i=l}^{n} \frac{P(Q^*) - c_i}{2g_i} + \sum_{i=1}^{n} \frac{P(Q^{\mathrm{U}}) - P(Q^*)}{g_i} \geq 0 \tag{26}$$

Where the equivalence follows from the fixed point equations (19) for Q^* and (20) for Q^{U}, respectively, for the case $u = m = n$.

The analysis will follow a mathematical programing approach. Specifically, we will show that any minimum of the left hand side of inequality (26), over the parameters of the problem (CAP) subject to the constraints defined by (22)-(25), $B \geq 0$ and $P(Q^{\mathrm{U}}) \geq P(Q^*)$, must be non-negative. We start by showing a preliminary result that allow us to simplify the analysis.

PROPOSITION 3. *For any given number of firms in the market $n \geq 2$, in order to study the worst-case performance of uniform co-payments there is no loss of generality in focusing on instances of the problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$. Moreover, when focusing on these instances condition (24) becomes redundant.*

The result in Proposition 3 allows us to simplify somewhat inequality (26) and rewrite it as

$$2Q^{\mathrm{U}} - Q^* \geq 0 \Leftrightarrow -\sqrt{\left(\sum_{i=l}^{n} \frac{P - c_i}{2g_i}\right)^2 + \sum_{i=l}^{n} \frac{B}{g_i} + \sum_{i=l}^{n} \frac{1}{g_i}\left(\sum_{i=l}^{n}\left(\frac{c_i - c_l}{2}\right)^2 \frac{1}{g_i}\right) - \left(\sum_{i=l}^{n} \frac{c_i - c_l}{2g_i}\right)^2}$$
$$2\sqrt{\left(\sum_{i=1}^{n} \frac{P - c_i}{2g_i}\right)^2 + \sum_{i=1}^{n} \frac{B}{g_i}} + \sum_{i=l}^{n} \frac{P - c_i}{2g_i} \geq 0. \tag{27}$$

Now we are ready to show the main result of this section.

THEOREM 1. *For any number $n \geq 2$ of price taking firms with affine increasing marginal cost $h(q_i) = c_i + g_i q_i$, where $c_i \geq 0$ and $g_i > 0$ for each firm $i \in \{1, \ldots, n\}$, facing a general decreasing inverse demand function $P(Q)$, where $\sum_{i=1}^{n} q_i = Q$. And for any budget $B \geq 0$, let Q^* be the market consumption induced by optimal co-payments, and Q^{U} be the market consumption induced by the uniform co-payments. Then $2Q^{\mathrm{U}} \geq Q^*$.*

Proof. From Propositions 2 and 3, and Lemma 2, it follows that it is enough to show that all instances of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$, and such that they satisfy conditions (22), (23), (25) and $B \geq 0$, must also satisfy the inequality (27).

Additionally, from Proposition 10 in Appendix A it follows that we only need to show that inequality (27) holds for all instances of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$, and at least one of the conditions (22), (23), (25) or $B \geq 0$ holds with equality. In particular, for any instance of problem (CAP) such that $B = 0$ we have that $Q^{\mathrm{U}} = Q^*$, and we are done in this case.

Moreover, from Propositions 11 and 12 in Appendix A it follows that we can reduce the proof to showing that inequality (27) holds for all instances of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$, and either both (22) and (25) hold with equality, or both (23) and (25) hold with equality.

For any $n \geq 2$, consider any instance of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$ and both (22) and (25) hold with equality. Note that this uniquely defines the value of P, for any n-dimensional vectors $\mathbf{c}$ and $\mathbf{g}$ where $0 \leq c_1 \leq \ldots \leq c_n \leq P(0)$, and $g_i > 0$ for each $i \in \{1, \ldots, n\}$, by

$$(c_n - P)\left(\sum_{i=1}^{n} \frac{c_n - c_i}{g_i}\right) = B = \sum_{i=l}^{n}\left(\frac{2P - c_i - c_l}{2}\right)\left(\frac{c_i - c_l}{2g_i}\right) \geq 0. \tag{28}$$

Note that from (28) it follows that $P < c_n$, and it also implies (23).

On the other hand, both (22) and (25) holding with equality is equivalent to $y_l^* = 0$ and $q_n^{\mathrm{U}} = 0$. From (18) for $m = n$ it follows that $y_l^* = 0$ implies

$$\sqrt{\left(\sum_{i=l}^{n} \frac{P - c_i}{2g_i}\right)^2 + \sum_{i=l}^{n} \frac{B}{g_i} + \sum_{i=l}^{n} \frac{1}{g_i} \left(\sum_{i=l}^{n} \left(\frac{c_i - c_l}{2}\right)^2 \frac{1}{g_i}\right) - \left(\sum_{i=l}^{n} \frac{c_i - c_l}{2g_i}\right)^2} = \sum_{i=l}^{n} \frac{P - c_l}{2g_i}. \tag{29}$$

Similarly, from equations (20) and (21) for $u = n$ it follows that $q_n^{\mathrm{U}} = 0$ implies

$$\sqrt{\left(\sum_{i=1}^{n} \frac{P - c_i}{2g_i}\right)^2 + \sum_{i=1}^{n} \frac{B}{g_i}} = \sum_{i=1}^{n} \frac{c_n - c_i}{g_i} + \sum_{i=1}^{n} \frac{c_n - P}{g_i}. \tag{30}$$

From (28), (29) and (30) it follows that if both (25) and (22) hold with equality, then the inequality (27) simplifies to

$$\sum_{i=l}^{n} \left(\frac{2P - c_i - c_l}{2}\right) \left(\frac{c_i - c_l}{2g_i}\right) + \sum_{i=1}^{n} \frac{(c_n - P)^2}{g_i} - (c_n - P) \sum_{i=l}^{n} \frac{c_i - c_l}{2g_i} \geq 0. \tag{31}$$

To conclude, note that for the value of P that satisfies condition (28) we have

$$\sum_{i=l}^{n} \left(\frac{2P - c_i - c_l}{2}\right) \frac{c_i - c_l}{2g_i} + \sum_{i=1}^{n} \frac{(c_n - P)^2}{g_i} - (c_n - P) \sum_{i=l}^{n} \frac{c_i - c_l}{2g_i} \geq \sum_{i=l}^{n} \left(\frac{2P - c_i - c_l}{2}\right) \frac{c_i - c_l}{2g_i} \geq 0.$$

Where the first inequality follows from the observation that the left hand side of inequality (31) is a convex function of P, and its minimizer P^* is such that $\sum_{i=1}^{n} \frac{c_n - P^*}{g_i} = \sum_{i=l}^{n} \frac{c_i - c_l}{2g_i}$. That is, the first inequality holds for P^*. Hence, it also holds for the value of P that satisfies condition (28). The second inequality follows from equation (28). This completes the proof when both (22) and (25) hold with equality. The case where both (23) and (25) hold with equality is analogous, and it is therefore omitted.

Theorem 3 shows that uniform co-payments are guaranteed to induce at least half of the market consumption induced by optimal co-payments in this model. This result is surprising, considering that this worst-case bound holds for *any* decreasing inverse demand function, for price taking producers with arbitrarily different affine increasing marginal cost functions. In particular, this result also applies to the relevant special case of Cournot competition with linear demand and affine marginal costs. It suggests that uniform co-payments are, despite their simplicity, a powerful policy tool to increase the market consumption of a good in a fairly general model.

One important remaining question is whether this bound can be improved for this model. We conclude this section by answering it in the negative, showing that the worst-case performance bound of Theorem 1 is asymptotically tight.

Proposition 4. *For any number of firms in the market $n \geq 2$, and for any parameters $c > 0$ and $\epsilon > 0$, with ϵ small enough, there exists a family of instances of problem (CAP) with budget $B(n,c)$, for price taking firms with affine increasing marginal costs $h_1(q_1) = (c-\epsilon)q_1$, $h_i(q_i) = c + (c-\epsilon)q_i$ for each $i \in \{2,\dots,n\}$, facing a linear decreasing inverse demand function, $P(Q) = a(n,c,\epsilon) - b(n,c,\epsilon)Q$, where $b(n,c,\epsilon) \xrightarrow{\epsilon\to 0} 0$, such that if we let Q^* and Q^{U} be the market consumption induced by optimal co-payments and uniform co-payments, respectively, then $\frac{Q^{\mathrm{U}}}{Q^*} \xrightarrow[\epsilon\to 0]{n\to\infty} \frac{1}{2}$. Namely, the market consumption induced by uniform co-payments is arbitrarily close to half of the market consumption induced by optimal co-payments.*

The details of this worst-case instance are described in the proof of Proposition 4 in Appendix A.

5. Additional Insights

In this section we derive additional insights on the performance of uniform co-payments in maximizing the market consumption of a good. In order to do this, we consider a mathematical programming relaxation of problem (CAP), where we ignore the non-negativity of the co-payments, as our benchmark. Note that this relaxation, that we denote (UBP), allows the central planner to charge per-unit taxes to the firms, as well as co-payments.

The first insight we obtain is that if the central planner is allowed to charge per-unit taxes to the firms, then the performance of uniform co-payments can be arbitrarily bad. Additionally, problem (UBP) allows us to connect our study of the performance of uniform co-payments in maximizing market consumption of a good with the literature on the price of anarchy in congestion games, see Roughgarden (2005). Specifically, from known and improved bounds on the total cost that is induced by selfish routing, we derive a worst-case performance guarantee of one half for uniform co-payments in maximizing market consumption in interesting settings where firms may have increasing non-linear marginal cost functions.

Because the latter results apply for non-linear increasing marginal cost functions $h_i(q_i)$, we define the mathematical relaxation of problem (CAP), where we ignore the non-negativity of the co-payments, for this more general setting.

$$
(UBP) \quad \begin{aligned}
\max_{\mathbf{q},Q} \;& Q \\
\text{s.}t. \;& \sum_{i=1}^{n} h_i(q_i)q_i - P(Q)Q \leq B && (32)\\
& \sum_{i=1}^{n} q_i = Q && (33)\\
& q_i \geq 0, \text{ for each } i \in \{1,\dots,n\}. && (34)
\end{aligned}
$$

5.1. Unbounded Performance when Taxes are Allowed

Proposition 5 below provides the main insight of this section.

PROPOSITION 5. *For any number of firms in the market $n \geq 2$, and for any parameter $\epsilon > 0$ small enough, there exists a family of instances of problem (CAP) with budget $B = \epsilon$, for price taking firms with affine increasing marginal costs $h_1(q_1) = c(n,\epsilon)q_1$, $h_i(q_i) = c(n,\epsilon)(1 + q_i)$ for each $i \in \{2,\ldots,n\}$, facing a linear decreasing inverse demand function, $P(Q) = a(n,\epsilon) - b(n,\epsilon)Q$, where $b(n,\epsilon) \xrightarrow{\epsilon\to 0} 0$, such that if we let Q^{UB} and Q^{U} be the market consumption induced in this instance of problem (UBP) by the optimal policy and by uniform co-payments, respectively, then $\frac{Q^{\mathrm{U}}}{Q^{\mathrm{UB}}} = \frac{2}{1+\sqrt{n-\epsilon}}$.*

The details of this instance of problem (UBP) are given in proof of Proposition 5 in the Online Appendix 2.1. Proposition 5 shows that if the central planner is allowed to charge per-unit taxes to the firms, then the performance of uniform co-payments can be arbitrarily bad as the number of firms in the market increases. This is in sharp contrast to the result of Theorem 1, which shows a guaranteed performance of one half for uniform co-payments when taxes are not allowed. These results suggest that implementing uniform co-payments may not be such a good idea if the central planner is in the position of charging taxes to the firms. However, in many practical cases the central planner role is played by an ONG, or by the World Bank, while the producers are firms operating in different countries, see for example AMFm Independent Evaluation Team (2012) for the case of anti-malarials described in the Introduction. In these cases, our results show that the efficiency loss incurred by implementing the simple and practical policy of uniform co-payments is bounded, and not too large.

Additionally, the instance from Proposition 5 is special in the following way. As ϵ approaches zero, it asymptotically matches the worst-case performance guarantee shown in Theorem 2 below, for price taking firms with affine increasing marginal costs that do not intersect, facing an arbitrary decreasing inverse demand function.

THEOREM 2. *For any number $n \geq 2$ of price taking firms with affine increasing marginal cost $h(q_i) = c_i + g_i q_i$, for each firm $i \in \{1,\ldots,n\}$, facing a general decreasing inverse demand function $P(Q)$, where $\sum_{i=1}^n q_i = Q$, $0 \leq c_1 \leq \ldots \leq c_n \leq P(0)$ and $0 < g_1 \leq \ldots \leq g_n$. And for any budget $B \geq 0$, let Q^{UB} be the market consumption induced by the optimal policy in problem (UBP), and Q^{U} be the market consumption induced by uniform co-payments. Then $\frac{Q^{\mathrm{U}}}{Q^{\mathrm{UB}}} \geq \frac{2}{1+\sqrt{n}}$.*

Proof. For any $n \geq 2$, and for any instance of problem (UBP) such that $h(q_i) = c_i + g_i q_i$ for each firm $i \in \{1,\ldots,n\}$, where $0 \leq c_1 \leq \ldots \leq c_n$ and $0 < g_1 \leq \ldots \leq g_n$, let $(\mathbf{q}^{\mathrm{UB}}, Q^{\mathrm{UB}})$ be the market outputs induced by the optimal solution of problem (UBP). From Lemma 5 in the Online Appendix 2.1 we have that at any optimal solution of problem (UBP), if $q_i^{\mathrm{UB}} > 0$ then $c_i + 2g_i q_i^{\mathrm{UB}} \geq c_j + 2g_i q_j^{\mathrm{UB}}$

for each firm $j \in \{1, \ldots, n\}$. It follows that $q_i^{\mathrm{UB}} > 0$ implies $q_j^{\mathrm{UB}} > 0$, for each $j \in \{1, \ldots, i\}$. Let index $m \in \{1, \ldots, n\}$ denote the last firm active in the market in this solution.

From essentially the same proof as in Proposition 2 it follows that, in order to study the worst-case performance of uniform co-payments, there is no loss of generality in focusing on instances of the problem (UBP) such that $m = u = n$. Hence, we assume this in the remainder of the proof.

Note that the market equilibrium condition (6) implies

$$Q^{\mathrm{U}} = \sum_{i=1}^{n} \frac{1}{g_i} \left(P(Q^{\mathrm{U}}) + \frac{B}{Q^{\mathrm{U}}} \right) - \sum_{i=1}^{n} \frac{c_i}{g_i}. \tag{35}$$

Similarly, Lemma 5 in the Online Appendix 2.1 implies

$$Q^{\mathrm{UB}} = \sum_{i=1}^{n} \frac{1}{g_i} \left(P(Q^{\mathrm{UB}}) + \frac{B}{Q^{\mathrm{UB}}} \right) - \sum_{i=1}^{n} \frac{c_i}{g_i} + \frac{1}{4Q^{\mathrm{UB}}} \left(\sum_{i=1}^{n} \frac{c_i^2}{g_i} \sum_{i=1}^{n} \frac{1}{g_i} - \left(\sum_{i=1}^{n} \frac{c_i}{g_i} \right)^2 \right). \tag{36}$$

To conclude, note that

$$Q^{\mathrm{UB}}(Q^{\mathrm{UB}} - Q^{\mathrm{U}}) \leq \frac{1}{4} \left(\sum_{i=1}^{n} \frac{c_i^2}{g_i} \sum_{i=1}^{n} \frac{1}{g_i} - \left(\sum_{i=1}^{n} \frac{c_i}{g_i} \right)^2 \right) \leq \frac{(n-1)}{4} \left(\sum_{i=1}^{n} \frac{c_n - c_i}{g_i} \right)^2 \leq \frac{(n-1)}{4} (Q^{\mathrm{UB}})^2. \tag{37}$$

Where the first inequality follows from equations (35), (36) and $Q^{\mathrm{UB}} \geq Q^{\mathrm{U}}$, the second inequality follows from Lemma 6 in the Online Appendix 2.1, and the third inequality follows from Lemma 1 for $u = n$. Note that (37) is equivalent to $\frac{Q^{\mathrm{UB}}}{Q^{\mathrm{U}}} \left(\frac{Q^{\mathrm{UB}}}{Q^{\mathrm{U}}} - 1 \right) \leq \frac{(n-1)}{4}$, which implies the inequality in the Theorem for any non-negative value of $\frac{Q^{\mathrm{UB}}}{Q^{\mathrm{U}}}$.

Theorem 2 shows that, as the linear inverse demand function approaches a constant, the instance given in Proposition 5 is the worst possible with respect to the performance of uniform co-payments in maximizing market consumption, for a fairly large family of models. On the positive side, Theorem 2 also shows that if the number of firms in the market is moderate, specifically nine or less, then the simple policy of allocating the same co-payment to each firm is a powerful alternative, even if the problem is faced by a government that could potentially also charge taxes in order to increase the market consumption of a good.

5.2. Results for Non-linear Increasing Marginal Costs

In this section, we make a connection between our study of the performance of uniform co-payments in maximizing market consumption, and the price of anarchy in congestion games. This connection allows us to get some insights into the performance of uniform co-payments in settings where the marginal cost functions of the firms may be non-linear.

Proposition 6. *For any number $n \geq 2$ of price taking firms with marginal cost $h_i(q_i)$ such that $h_i(q_i)q_i$ is convex for each firm $i \in \{1, \ldots, n\}$, facing a general decreasing inverse demand function*

$P(Q)$, where $\sum_{i=1}^{n} q_i = Q$. And for any budget $B \geq 0$, let Q^{UB} be the market consumption induced by the optimal policy in problem (UBP), and Q^{U} be the market consumption induced by uniform co-payments. Then

$$\text{If } 2Q^{\mathrm{U}} \leq Q^{\mathrm{UB}} \text{ then } P(Q^{\mathrm{U}})Q^{\mathrm{U}} \leq P(Q^{\mathrm{UB}})Q^{\mathrm{UB}}. \tag{38}$$

Moreover, if we assume that the marginal cost functions $h(q_i)$ are additionally concave for each firm $i \in \{1, \ldots, n\}$, then

$$\text{If } 2Q^{\mathrm{U}} \leq Q^{\mathrm{UB}} \text{ then } \frac{3}{2}P(Q^{\mathrm{U}})Q^{\mathrm{U}} + \frac{B}{2} \leq P(Q^{\mathrm{UB}})Q^{\mathrm{UB}}. \tag{39}$$

The proof of equation (38) follows from an application of Theorem 3.1 in Roughgarden and Tardos (2002), while the proof of equation (39) follows from an improvement of this result for the special case of concave increasing marginal cost functions, see the Online Appendix 2.1.

First, note that equation (38) implies that for any decreasing inverse demand function $P(Q)$ such that the total revenue function $P(Q)Q$ is also decreasing, uniform co-payments are guaranteed to induce at least half of the market consumption induced by the optimal policy of problem (UBP), for any number of price taking firms with increasing marginal cost $h_i(q_i)$, such that $h_i(q_i)q_i$ is convex. Similarly, for the same conditions on the firms' marginal cost functions, Equation (38) implies that for any decreasing $P(Q)$ such that $P(Q)Q$ is unimodal, if the market consumption induced by uniform co-payments Q^{U} is larger than the maximizer of $P(Q)Q$, Q^{M}, then uniform co-payments are guaranteed to induce at least half of the market consumption induced by the optimal policy of problem (UBP). By the same logic, equation (39) implies that if the marginal cost functions $h(q_i)$ are additionally concave, then Q^{U} only needs to be close enough to Q^{M} for the one half guarantee to hold, depending on the specific shape of the total revenue function $P(Q)Q$, and the budget B, as in equation (39). Again, this worst-case performance guarantee is with respect to the best policy that can charge per-unit taxes to the producers, as well as allocating co-payments.

This insight is relevant because the regime where the total revenue function $P(Q)Q$ is decreasing is precisely the area where subsidies are needed the most in order to increase the market consumption of a good. This is because no producer has the incentive to produce more when the total market output attains Q^{M}, which is the quantity that would be produced by a monopolist with no production costs. Proposition 6 shows that if the market consumption induced by uniform co-payments is in, or close enough to, this regime, then uniform co-payments have a worst-case performance guarantee of one half. Moreover, this result provides a plausible explanation for the good performance of uniform co-payments observed in numerical simulations in several settings with a general decreasing inverse demand function, and increasing non-linear marginal costs, that satisfy the conditions of Proposition 6, see The authors (2014).

6. A Better Bound for an Important Special Case

In this section, we consider the important case of Cournot competition with linear demand and constant marginal costs. As discussed in Section 2, this is an important model frequently used by researchers as a building block to get detailed insights into complex phenomena. Moreover, the market output in this market equilibrium is equivalent to the one attained by price taking firms with marginal cost functions $h_i(q_i) = c_i + bq_i$ for each firm $i \in \{1,\dots,n\}$, facing a linear inverse demand function $P(Q) = a - bQ$. The goal in this section will be to show that, for any given number of firms in the market $n \geq 2$, any instance of problem (CAP) specialized to this model must be such that

$$\frac{Q^{\mathrm{U}}}{Q^*} \geq \frac{2+\sqrt{2+2/n}}{4} \geq \frac{2+\sqrt{2}}{4} \approx 85.36\%.$$

This provides a surprisingly high worst-case performance guarantee for uniform co-payments in maximizing the market consumption of a good in this model, for each $n \geq 2$. Moreover, we will additionally show that this guarantee is *tight* and cannot be improved. This result follows a more detailed analysis than Theorem 1, in the sense that it provides tight insights into the worst case performance of uniform co-payments for any finite number of firms. However, its proof is also more involved. For this reason we will restrict ourselves to a proof sketch here, and present a more detailed proof structure in the Online Appendix 2.2.

We start by showing that for this model we can provide *closed form expressions* for the market consumption induced by optimal co-payments and uniform co-payments, respectively.

PROPOSITION 7. *For the special case of $n \geq 2$ Cournot competitors with constant marginal costs $c_i \geq 0$, for each firm $i \in \{1,\dots,n\}$, facing a linear inverse demand function $P(Q) = a - bQ$, for $a > 0$, $b > 0$, the quantities in Proposition 1 and Lemma 1 can be further specified as follows.*

$$q_l^* = \frac{a + \sum_{i=1}^m c_i}{(m+1)b} - \frac{c_l}{b} - \sum_{i=l+1}^{m} \frac{c_i - c_l}{2(m+1)b} + \frac{l}{(m+1)b} y_l^*. \tag{40}$$

$$\begin{aligned} y_l^* = &-\left(\frac{a+\sum_{i=1}^{l-1} c_i - lc_l}{2l}\right) + \left(\left(\frac{a+\sum_{i=1}^{l-1} c_i - lc_l}{2l}\right)^2 + \frac{m+1}{(m-l+1)l}\sum_{i=l}^{m}\frac{(c_i-c_l)^2}{4} + \frac{m+1}{(m-l+1)l} bB \right.\\ &\left. - \sum_{i=l}^{m} \frac{c_i - c_l}{2}\left(a + \sum_{i=1}^{l-1} c_i + \sum_{i=l+1}^{m}\frac{c_i}{2} - (m+l)\frac{c_l}{2}\right)\frac{1}{(m-l+1)l}\right)^{1/2}. \end{aligned} \tag{41}$$

$$\begin{aligned} Q^* = &\frac{1}{2(m+1)lb}\left((2lm - m + l - 1)a - (m+l+1)\sum_{i=1}^{l-1} c_i - lc_l - l\sum_{i=l+1}^{m} c_i\right) \\ &+ \frac{m-l+1}{2(m+1)lb}\left(\left(a + \sum_{i=1}^{l-1} c_i - lc_l\right)^2 + \frac{l(m+1)}{m-l+1}\sum_{i=l+1}^{m}(c_i - c_l)^2 + \frac{4l(m+1)}{m-l+1} bB\right. \end{aligned}$$

$$-\frac{l}{m-l+1}\left(\sum_{i=l+1}^{m}(c_i-c_l)\right)\left(2a+2\sum_{i=1}^{l-1}c_i+\sum_{i=l+1}^{m}c_i-(m+l)c_l\right)\Bigg)^{1/2}. \qquad (42)$$

$$Q^{\mathrm{U}}=\frac{ua-\sum_{i=1}^{u}c_i+\sqrt{\left(ua-\sum_{i=1}^{u}c_i\right)^2+4u(u+1)bB}}{2(u+1)b}. \qquad (43)$$

Moreover, if the index l defined in Proposition 1 satisfies $l\geq 2$, then

$$Q^*\leq\frac{2ma-2\sum_{i=1}^{l-2}c_i-(m-l+3)c_{l-1}-\sum_{i=l}^{m}c_i}{2(m+1)b}. \qquad (44)$$

This bound is attained when $y_l^*=\frac{c_l-c_{l-1}}{2}$.

The closed form expressions given in Proposition 7 further specify the characterization of the instances of problem (CAP) that satisfy the condition $l\leq m=u=n$. Recall, from Proposition 2, that there is no loss of generality in focusing on them in order to study the worst-case performance of uniform co-payments. Therefore, we update Lemma 2 accordingly.

LEMMA 3. *For the special case of $n\geq 2$ Cournot competitors with constant marginal costs $c_i\geq 0$, for each firm $i\in\{1,\dots,n\}$, facing a linear inverse demand function $P(Q)=a-bQ$, for $a>0$, $b>0$. Let $l,m,u\in\{1,\dots,n\}$ be indexes as defined in Proposition 1 and Lemma 1, respectively. Then, for any given number of firms in the market $n\geq 2$, an instance of problem (CAP) in this special case is such that $l\leq m=u=n$ if and only if it satisfies the following conditions*

$$y_l^*\geq 0\Leftrightarrow bB+\sum_{i=l+1}^{n}\frac{(c_i-c_l)^2}{4}-\sum_{i=l+1}^{n}\frac{c_i-c_l}{4(n+1)}\left(2a+2\sum_{i=1}^{l-1}c_i+\sum_{i=l+1}^{n}c_i-(n+l)c_l\right)\geq 0. \qquad (45)$$

$$y_l^*\leq\frac{c_l-c_{l-1}}{2}\Leftrightarrow bB+\sum_{i=l}^{n}\frac{(c_i-c_{l-1})^2}{4}-\sum_{i=l}^{n}\frac{c_i-c_{l-1}}{4(n+1)}\left(2a+2\sum_{i=1}^{l-1}c_i+\sum_{i=l}^{n}c_i-(n+l)c_{l-1}\right)\leq 0. \qquad (46)$$

$$\begin{aligned}q_n^*\geq 0\Leftrightarrow bB&+\sum_{i=l+1}^{n}\frac{(c_i-c_l)^2}{4}-\sum_{i=l+1}^{n}\frac{c_i-c_l}{4(n+1)}\left(2a+2\sum_{i=1}^{l-1}c_i+\sum_{i=l+1}^{n}c_i-(n+l)c_l\right)\\&+\frac{n-l+1}{4l(n+1)}\left(\left(a+\sum_{i=1}^{l-1}c_i-lc_l\right)^2-\left((n+1)c_n-\sum_{i=1}^{n}c_i-a\right)^2\right)\geq 0.\end{aligned} \qquad (47)$$

$$q_n^{\mathrm{U}}\geq 0\Leftrightarrow bB-\left(nc_n-\sum_{i=1}^{n}c_i\right)\left((n+1)c_n-\sum_{i=1}^{n}c_i-a\right)\geq 0. \qquad (48)$$

The proofs of Proposition 7 and Lemma 3 are analogous to Proposition 1 and Lemma 2, and are omitted. We continue with an observation that allows to simplify the analysis. Proposition 8 below shows that, without loss of generality, in order to study the worst-case performance of uniform co-payments in this special case, we can focus on instances of problem (CAP) such that, in the market equilibrium induced by the uniform co-payments, the last firm in the market is exactly on the verge of start having a positive market output. This is important because it significantly simplifies the expression for the market consumption induced by uniform co-payments.

PROPOSITION 8. *For any given number $n \geq 2$ of Cournot competitors with constant marginal costs $c_i \geq 0$, for each firm $i \in \{1,\dots,n\}$, facing a linear decreasing inverse demand function $P(Q) = a - bQ$, where $\sum_{i=1}^n q_i = Q$, $a > 0$ and $b > 0$. And for any budget $B \geq 0$, let Q^* be the market consumption induced by optimal co-payments, and Q^{U} be the market consumption induced by uniform co-payments. Then, there exists an instance of problem (CAP) that minimizes the ratio Q^{U}/Q^*. Moreover, any instance of problem (CAP) that minimizes the ratio Q^{U}/Q^* must be such that $q_n^{\mathrm{U}} = 0$. Namely, such that condition (48) holds with equality and*

$$Q^{\mathrm{U}} = \frac{nc_n - \sum_{i=1}^n c_i}{b}. \tag{49}$$

The proof of Proposition 8 is presented in the Online Appendix 2.2.

Our goal now is to introduce a mathematical program whose optimal solution quantifies the worst-case performance of uniform co-payments in maximizing the aggregated market consumption of a good, for the special case of Cournot competition with linear demand and constant marginal costs. We will show that for any given number of firms in the market $n \geq 2$, the instance of problem (CAP) that minimizes the ratio Q^{U}/Q^* will be defined by the marginal costs c_i, and the budget B, for any given demand parameter values $a > 0$, $b > 0$. Moreover, the value of the worst-case ratio will be independent of the values of a, b. Therefore, we will only consider c_i, for each $i \in \{1,\dots,n\}$, and B as variables, while we will treat $a > 0$ and $b > 0$ as parameters.

To simplify the notation let us define the function

$$\begin{aligned} &\sqrt{*_l}(B,\mathbf{c}) \\ &\equiv (n-l+1)\left(\left(a + \sum_{i=1}^{l-1} c_i - lc_l\right)^2 + \frac{l(n+1)}{n-l+1}\sum_{i=l+1}^{n}(c_i - c_l)^2 + \frac{4l(n+1)}{n-l+1}bB \right. \\ &\left. -\frac{l}{n-l+1}\left(\sum_{i=l+1}^{n}(c_i - c_l)\right)\left(2a + 2\sum_{i=1}^{l-1} c_i + \sum_{i=l+1}^{n} c_i - (n+l)c_l\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{50}$$

Then, the following lemma defines the main component of the problem we are interested in solving.

LEMMA 4. *For any given number $n \geq 2$ of Cournot competitors with constant marginal costs $c_i \geq 0$, for each firm $i \in \{1,\dots,n\}$, facing a linear decreasing inverse demand function $P(Q) = a - bQ$, where $\sum_{i=1}^n q_i = Q$, $a > 0$ and $b > 0$, let $WC(l,n)$ be the worst-case performance of uniform* *co-payments in maximizing the market consumption, assuming that the index* $l \in \{1,\dots,n\}$ *is fixed. Then,* $WC(l,n)$ *can be computed as the optimal objective value of problem* (WCP_l) *below*

$$\begin{aligned}
& WC(l,n) \equiv \\
\min_{B,c} \quad & \frac{Q^{\mathrm{U}}(\boldsymbol{c})}{Q_l^*(B,\boldsymbol{c})} = \frac{2(n+1)l\,(nc_n - \sum_{i=1}^n c_i)}{(2ln-n+l-1)a-(n+l+1)\sum_{i=1}^{l-1}c_i - l\sum_{i=l}^n c_i + \sqrt{*_l}(B,\boldsymbol{c})} \\
s.t. \quad & 0 \le c_1 && (51)\\
& c_i \le c_{i+1}, \ \textit{for each } i \in \{1,\dots,n-1\} && (52)\\
(WCP_l) \quad & c_n \le a && (53)\\
& B \ge 0 && (54)\\
& \textit{and conditions } (45),(46),(47),(48). && (55)
\end{aligned}$$

To simplify the notation of $WC(l,n)$, we have omitted its dependence on the demand parameters a and b. This is because the worst-case ratio will be independent of their values. Note that the expression for Q^{U} follows from Proposition 8, while the expression for Q^* follows from equation (42) in Proposition 7, taking $u = m - n$.

A priori, for any given number of firms in the market $n \ge 2$, and demand parameters $a > 0$, $b > 0$, it is not clear which case of $l \in \{1,\dots,n\}$ attains the worst-case performance of uniform co-payments in maximizing the market consumption in this setting. Therefore, we are interested in solving the following problem

$$\min_{l \in \{1,\dots,n\}} WC(l,n). \tag{56}$$

Now we are ready to state the main result in this section.

Theorem 3. *For any given number* $n \ge 2$ *of Cournot competitors with constant marginal costs* $c_i \ge 0$*, for each firm* $i \in \{1,\dots,n\}$*, facing a linear decreasing inverse demand function* $P(Q) = a - bQ$*, where* $\sum_{i=1}^n q_i = Q$*,* $a > 0$ *and* $b > 0$*. And for any budget* $B \ge 0$*, let* Q^* *be the market consumption induced by optimal co-payments, and* Q^{U} *be the market consumption induced by uniform co-payments. Then*

$$\frac{Q^{\mathrm{U}}}{Q^*} \ge \frac{2+\sqrt{2+2/n}}{4} \ge \frac{2+\sqrt{2}}{4} \approx 85.36\%. \tag{57}$$

Proof. We equivalently show that for any $n \ge 2$

$$\min_{l \in \{1,\dots,n\}} WC(l,n) = \frac{2+\sqrt{2+2/n}}{4}. \tag{58}$$

The proof outline is as follows. First, we give a proof sketch that shows that, for any $n \ge 2$, $a > 0$, $b > 0$, the minimum in problem (58) must be attained for the case $l = 2$. Then, we conclude by giving a proof sketch that shows that $WC(2,n) = \frac{2+\sqrt{2+2/n}}{4}$.

We want to show that, for any given number of firms in the market $n \geq 2$, and demand parameters $a > 0$, $b > 0$, we have that

$$WC(2,n) \leq WC(l,n), \text{ for each } l \in \{1,\dots,n\}. \tag{59}$$

Proposition 13 in the Online Appendix 2.2 shows that, for any optimal solution to problem (WCP_1), condition (45) must be binding. This implies $WC(2,n) \leq WC(1,n)$.

Similarly, Proposition 15 in the Online Appendix 2.2 shows that, for any given $n \geq 3$, $a > 0$, $b > 0$, we must have that $WC(3,n) \geq \frac{2+\sqrt{2+2/n}}{4} \geq WC(2,n)$, where the last inequality follows from the fact that the right hand side is attained by the candidate instance for the case $l = 2$ from Proposition 9 in Section 6.1 below.

Additionally, Proposition 14 in the Online Appendix 2.2 provides a parametric lower bound, based on linear programming, on $WC(l,n)$ for any $n \geq 2$ and for each $l \in \{2,\dots,n\}$. This lower bound implies that for any given $n \geq 4$, $a > 0$, $b > 0$ and for each $l \in \{4,\dots,n\}$, we have that

$$WC(l,n) \geq \frac{2nl - 2n + 2l - 2}{2nl - n + l - 1} \geq \frac{6(n+1)}{7n+3} \geq \frac{2+\sqrt{2+2/n}}{4} \geq WC(2,n),$$

where the first inequality follows from Proposition 14, the second inequality follows from the left hand side being increasing in l (the numerator increases faster than the denominator), and taking $l = 4$. The third inequality holds for any $n \geq 1$. The last inequality follows from the fact that the left hand side is attained by the candidate instance from Proposition 9, for the case $l = 2$. This completes the proof of inequality (59).

To conclude, we show that for any number of firms in the market $n \geq 2$, and demand parameters $a > 0$, $b > 0$, we have that $WC(2,n) = \frac{2+\sqrt{2+2/n}}{4}$.

Specifically, for any given $n \geq 2$, $a > 0$, $b > 0$, Proposition 16 in the Online Appendix 2.2 describes a single variable mathematical programming relaxation of problem (WCP_2), denoted by $(RWCP_{2,1})$, whose optimal solution provides a lower bound on $WC(2,n)$. We will show that the candidate instance from Proposition 9 in Section 6.1 is the optimal solution to the relaxation $(RWCP_{2,1})$. Because this instance is in fact feasible for the original problem (WCP_2), it follows that it is optimal for this problem as well.

It can be showed that the objective function of problem $(RWCP_{2,1})$ is quasiconvex in c_n. Now we show that the candidate instance from Proposition 9 is its unique minimizer. Any interior stationary solution must satisfy $\frac{d\, Q^{\mathrm{U}}(c_n^*) / Q_{2,1}^*(c_n^*)}{dc_n} = 0$. After simplifying, this condition is equivalent to

$$\sqrt{(n-1)(a-2c_n^*)\left((n-1)a - 2(5n+3)c_n^*\right)} = \frac{2(n-1)(3n+1)c_n^* - (n-1)^2 a}{3n+1}. \tag{60}$$

Equation (60) is quadratic in c_n^*, and its unique non-negative solution is $c_n^* = \left(\frac{n+\sqrt{\frac{n(n+1)}{2}}}{3n+1}\right) a \in \left[\frac{a}{2}, a\right]$, where $\left[\frac{a}{2}, a\right]$ is the feasible set of problem $(RWCP_{2,1})$. Hence, we conclude that this is the unique minimizer of $(RWCP_{2,1})$. This completes the proof sketch.

Theorem 3 shows that the efficiency loss in maximizing the market consumption induced by implementing the much simpler uniform co-payments policy is at most 15%, for any instance of Cournot competition with linear demand and constant marginal costs. Hence, the practical advantages presented by the uniform co-payments -including ease of implementation, communication and control of the co-payments program- should be weighted against this bounded efficiency loss. Furthermore, note that for any finite number of firms in the market n, the guarantee is strictly larger than this uniform bound. In particular, if the number of firms in the market is $n \in \{2, 3\}$, then uniform co-payments are guaranteed to induce more than 90% of the optimal market consumption.

6.1. Worst-case Instance

We conclude this section by emphasizing that, for any given number $n \geq 2$ of firms in the market, the worst-case performance bound of Theorem 3 is tight. That is, there exists a family of instances of problem (CAP), for Cournot competitors with constant marginal costs facing a linear decreasing inverse demand function, such that the market consumption induced by uniform co-payments is arbitrarily close to $\frac{2+\sqrt{2}}{4} \approx 85.36\%$ of the market consumption induced by optimal co-payments, as the number of firms in the market grows to infinity.

PROPOSITION 9. *For any number $n \geq 2$ of Cournot competitors with constant marginal costs, facing any inverse demand function $P(Q) = a - bQ$, where $a > 0$, $b > 0$, there exists a family of instances of problem (CAP) defined by the marginal cost functions $c_1 = 0$, $c_i = c(n, a) > 0$ for each $i \in \{2, \ldots, n\}$, and the budget $B(n, a, b)$, such that if we let Q^* and Q^{U} be the market consumption induced by optimal co-payments and uniform co-payments, respectively, then $\frac{Q^{\mathrm{U}}}{Q^*} = \frac{2+\sqrt{2+2/n}}{4} \xrightarrow{n\to\infty} \frac{2+\sqrt{2}}{4}$.*

The details of this instance are given in the proof of Proposition 9 in the Online Appendix 2.2.

Note that the value of the ratio $\frac{Q^{\mathrm{U}}}{Q^*}$ induced by this instance is independent of the values of the demand parameters a and b, and it matches the worst case bound from Theorem 3.

7. Conclusions

We study the problem faced by a central planner allocating a budget in the form of co-payment subsidies to heterogeneous competing producers of a commodity, with the goal of maximizing its aggregated market consumption. The policy that is most frequently implemented in practical applications of this problem is uniform, in the sense that every firm gets the same co-payment, even if some firms may be significantly more efficient than others. The central question of this paper is to evaluate the worst possible efficiency loss of uniform co-payments, when compared to the optimal firm-dependent co-payment allocation.

We present the first *worst-case guarantees* for the performance of uniform co-payments in such a model, which provides theoretical support for the use of this simple policy in practice. Specifically, we show that uniform co-payments are guaranteed to induce *at least half* of the market consumption induced by optimal co-payments, in a fairly general setting. Furthermore, for important special cases we show an improved guarantee of more than 85% of the optimal market consumption.

In summary, our results suggest that decision makers facing these type of co-payments allocation problems, should not spend time and resources looking into, and collecting the information to support, more sophisticated co-payment allocation policies, as the very simple uniform co-payments policy is likely to provide most of their benefits.

Future research on this topic should study whether the worst-case bounds for the performance of uniform co-payments presented in this paper hold for a larger family of instances, as well as whether there are generalized worst-case bounds that show that uniform co-payments have a guaranteed performance for additional market competition models.

Appendix A: Proofs of Section 4

Proof of Proposition 2

Proof. For any $n \geq 2$, consider an instance of problem (CAP) and let $l, m, u \in \{1, \ldots, n\}$ be the indexes defined in Proposition 1 and Lemma 1, respectively. First, assume that $u < m$. We will show that there exists a modified instance of problem (CAP) that attains a strictly worst performance of uniform co-payments.

Let $(\mathbf{q}^{\mathrm{U}}, Q^{\mathrm{U}})$ be the solution induced by uniform co-payments in this instance. Then, at the market equilibrium induced by uniform co-payments we must have $c_u + g_u q_u^{\mathrm{U}} = P(Q^{\mathrm{U}}) + \frac{B}{Q^{\mathrm{U}}} < c_m$. Let $\hat{i}$ be the first index such that $\hat{i} \geq l$, and $c_u < c_{\hat{i}}$. Note that $\hat{i} \in \{l, \ldots, m\}$, then $c_{\hat{i}} > P(Q^{\mathrm{U}}) + \frac{B}{Q^{\mathrm{U}}}$. It follows that we can reduce the value of $c_{\hat{i}}$, by $\epsilon > 0$ sufficiently small, without affecting the uniform co-payments solution $(\mathbf{q}^{\mathrm{U}}, Q^{\mathrm{U}})$., while obtaining a strictly larger value for the aggregated consumption induced by optimal co-payments. Specifically, let $(\mathbf{q}^*, Q^*)$ be an optimal solution to problem (CAP) for the original instance where $u < m$. Note that $(\mathbf{q}^*, Q^*)$ is feasible for the modified instance where we reduce the value of $c_{\hat{i}}$, by $\epsilon > 0$ sufficiently small. Moreover, from the observation that the budget constraint is not binding for $(\mathbf{q}^*, Q^*)$ in the modified instance, it follows that we can increase some q_i^*, and Q^*, by $\delta > 0$ small enough, maintain the feasibility for problem (CAP), and obtain a strictly larger objective value, strictly reducing the relative performance of uniform co-payments.

Now assume that $u > m$. We will again show that there exists a modified instance of problem (CAP) that attains a strictly smaller value of the ratio Q^{U}/Q^*. Note that this implies $c_u > c_m$ and $q_u^{\mathrm{U}} > 0$. Therefore, we can increase the value of c_u by $\epsilon > 0$ sufficiently small, without changing the optimal co-payments solution $(\mathbf{q}^*, Q^*)$, while decreasing Q^{U}. Specifically, from $u = n$ and $q_u^{\mathrm{U}} > 0$ it follows that the set of active firms at the market equilibrium will not change. Additionally, the derivatives of the right hand side, and left hand side, of equation (20) with respect to c_u is, respectively

$$-\frac{1}{2g_u}-\frac{\frac{1}{2g_u}\sum_{i=1}^{u}\frac{P(Q^{\mathrm{U}})-c_i}{2g_i}}{\sqrt{\left(\sum_{i=1}^{u}\frac{P(Q^{\mathrm{U}})-c_i}{2g_i}\right)^2+\sum_{i=1}^{u}\frac{B}{g_i}}}<0,\text{ and }\sum_{i=1}^{u}\frac{P'(Q^{\mathrm{U}})}{2g_i}+\frac{\sum_{i=1}^{u}\frac{P'(Q^{\mathrm{U}})}{2g_i}\sum_{i=1}^{u}\frac{P(Q^{\mathrm{U}})-c_i}{2g_i}}{\sqrt{\left(\sum_{i=1}^{u}\frac{P(Q^{\mathrm{U}})-c_i}{2g_i}\right)^2+\sum_{i=1}^{u}\frac{B}{g_i}}}<0.$$

Hence, Q^{U} decreases, strictly reducing the relative performance of uniform co-payments with respect to the original instance.

Proof of Proposition 3

Proof. Note that the left hand side of equation (26) is increasing in $P(Q^{\mathrm{U}})$ and decreasing in $P(Q^*)$. Additionally, the left hand side of condition (25) is increasing in $P(Q^{\mathrm{U}})$, while the left hand side of condition (22) is decreasing, and the left hand side of conditions (23) and (24) are increasing, in $P(Q^*)$. Hence, in order to study the worst-case performance of uniform co-payments there is no loss of generality in focusing on instances of the problem (CAP) such that either $P(Q^{\mathrm{U}})=P(Q^*)=P$, or both conditions (25) and (22) hold with equality. To conclude, we will show that in the latter case there is no loss of generality in focusing on the case where $P(Q^{\mathrm{U}})=P(Q^*)=P$ either.

Consider any instance of the problem (CAP) such that both conditions (25) and (22) hold with equality. Namely, such that $q_n^{\mathrm{U}}=0$ and $y_l^*=0$. From equation (18) for $m=n$ it follows that $y_l^*=0$ implies

$$\sqrt{\left(\sum_{i=l}^{n}\frac{P(Q^*)-c_i}{2g_i}\right)^2+\sum_{i=l}^{n}\frac{B}{g_i}+\sum_{i=l}^{n}\frac{1}{g_i}\left(\sum_{i=l}^{n}\left(\frac{c_i-c_l}{2}\right)^2\frac{1}{g_i}\right)-\left(\sum_{i=l}^{n}\frac{c_i-c_l}{2g_i}\right)^2}=\sum_{i=l}^{n}\frac{P(Q^*)-c_l}{2g_i}. \tag{61}$$

Similarly, from equations (20) and (21) for $u=n$ it follows that $q_n^{\mathrm{U}}=0$ implies

$$\sqrt{\left(\sum_{i=1}^{n}\frac{P(Q^{\mathrm{U}})-c_i}{2g_i}\right)^2+\sum_{i=1}^{n}\frac{B}{g_i}}=\sum_{i=1}^{n}\frac{2c_n-P(Q^{\mathrm{U}})-c_i}{2g_i}. \tag{62}$$

From (61) and (62) it follows that if both (25) and (22) hold with equality, then (26) simplifies to $\sum_{i=1}^{n}\frac{2c_n-P(Q^*)-c_i}{g_i}-\sum_{i=l}^{n}\frac{c_i-c_l}{2g_i}\geq 0$, where the left hand side is decreasing in $P(Q^*)$ and independent of $P(Q^{\mathrm{U}})$. Moreover, that both (25) and (22) hold with equality implies $\left(c_n-P(Q^{\mathrm{U}})\right)\left(\sum_{i=1}^{n}\frac{c_n-c_i}{g_i}\right)=B=\sum_{i=l}^{n}\left(\frac{2P(Q^*)-c_i-c_l}{2}\right)\left(\frac{c_i-c_l}{2g_i}\right)\geq 0$. Namely, in order the worst-case performance of uniform co-payments, if both (25) and (22) hold with equality and $P(Q^{\mathrm{U}})>P(Q^*)$, then we can always increase $P(Q^*)$ and decrease $P(Q^{\mathrm{U}})$ until $P(Q^{\mathrm{U}})=P(Q^*)=P$.

To conclude note that in this case (25) implies (24). Specifically, $B\geq 0$ implies $c_n\geq P$, and from (25) and (24) it is enough to show $(c_n-P)\left(\sum_{i=1}^{n}\frac{c_n-c_i}{g_i}\right)\geq\sum_{i=l}^{n}\left(\frac{c_n-c_i}{2}\right)\left(\frac{c_n+c_i-2P}{2g_i}\right)$, which holds for any $c_n\geq P$. This completes the proof.

Proposition 10. *For any given number of firms in the market $n\geq 2$, in order to study the worst-case performance of uniform co-payments there is no loss of generality in focusing on instances of the problem (CAP) such that $P(Q^{\mathrm{U}})=P(Q^*)=P$ and at least one of the conditions (22), (23), (25) or $B\geq 0$ holds with equality.*

Proof. From Proposition 3 it follows that we can focus, without loss of generality, on instances such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$. Consider any n-dimensional vectors $\mathbf{c}$ and $\mathbf{g}$ where $0 \le c_1 \le \ldots \le c_n \le P(0)$, and $g_i > 0$ for each $i \in \{1, \ldots, n\}$, such that the set of values of (B, P) that satisfy conditions (22), (23), (25) and $B \ge 0$ is non-empty. We show that the minimum over (B, P) of the left hand side of inequality (27) must satisfy that at least one of (22), (23), (25) or $B \ge 0$ holds with equality.

Denote by $f(B, P)$ the function defined by the left hand side of inequality (27), and note that $f(B, P)$ is continuous and coercive. Additionally, note that conditions (22), (23), (25) and $B \ge 0$ define a closed feasible set for (B, P), ensuring the existence of a minimum of the function $f(B, P)$, see for example Bertsekas (1999).

Note that a minimum cannot be attained at an interior point of the feasible set. Specifically, we have that $\frac{\partial f(B,P)}{\partial B} = 0$ implies $\frac{\partial f(B,P)}{\partial B} > 0$. Therefore, the minimum must be attained at a point (B, P) such that at least one constraint of the feasible set is binding, or equivalently at least one of (22), (23), (25) or $B \ge 0$ holds with equality.

Proposition 11. *For any given number of firms in the market $n \ge 2$, any instance of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$ and condition (22), or condition (23), holds with equality is dominated, in terms of the worst-case performance of uniform co-payments, by an instance of problem (CAP) where additionally condition (25) also holds with equality.*

Proof. For any $n \ge 2$, consider any n-dimensional vectors $\mathbf{c}$ and $\mathbf{g}$ where $0 \le c_1 \le \ldots \le c_n \le P(0)$, and $g_i > 0$ for each $i \in \{1, \ldots, n\}$, such that the set of values of (B, P) that satisfy condition (22) with equality, and conditions (23), (25) and $B \ge 0$, is non-empty. We show that any minimum over P of the left hand side of inequality (27) must also satisfy condition (25) with equality.

Note that from condition (22) holding with equality we can rewrite condition (25) as

$$P \ge \frac{c_n \sum_{i=1}^{n} \frac{c_n - c_i}{g_i} + \sum_{i=l}^{n} \left(\frac{c_i + c_l}{2}\right)\left(\frac{c_i - c_l}{2gi}\right)}{\sum_{i=1}^{n} \frac{c_n - c_i}{g_i} + \sum_{i=l}^{n} \frac{c_i - c_l}{2gi}}, \tag{63}$$

Which implies $P \ge c_l \ge 0$, as well as condition (23) and $B \ge 0$.

Additionally, consider any instance of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$ and condition (22) holds with equality, that is $y_l^* = 0$. From equation (18) for $m = n$, together with the expression for B from (22) with equality, it follows that the inequality (27) simplifies to

$$2\sqrt{\left(\sum_{i=1}^{n} \frac{P - c_i}{2g_i}\right)^2 + \sum_{i=1}^{n} \frac{1}{g_i} \sum_{i=l}^{n} \left(\frac{2P - c_i - c_l}{2g_i}\right)\left(\frac{c_i - c_l}{2}\right)} - \sum_{i=l}^{n} \frac{c_i - c_l}{2g_i} \ge 0 \tag{64}$$

Hence, the proof reduces to showing that any minimum of the left hand side of equation (64) must be attained (63), or equivalently (25), also holds with equality.

Denote by $f(P)$ the function defined by the left hand side of inequality (64), and note that $f(P)$ is continuous and increasing. Therefore, it attains its minimum at the lower bound of the interval defined by equation (63), or equivalently, when (25) also holds with equality. The proof for the case where condition (23) holds with equality is analogous, and it is therefore omitted.

PROPOSITION 12. *For any given number of firms in the market $n \geq 2$, any instance of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$ and condition (25) holds with equality is dominated, in terms of the worst-case performance of uniform co-payments, by an instance of problem (CAP) where additionally at least one of the conditions (22) or (23) also holds with equality.*

Proof. For any $n \geq 2$, consider any n-dimensional vectors $\mathbf{c}$ and $\mathbf{g}$ where $0 \leq c_1 \leq \ldots \leq c_n \leq P(0)$, and $g_i > 0$ for each $i \in \{1, \ldots, n\}$, such that the set of values of (B, P) that satisfy condition (25) with equality, and conditions (22), (23) and $B \geq 0$, is non-empty. We will show that any minimum over P of the left hand side of inequality (27) must also satisfy that at least one of (22) or (23) holds with equality.

Note that from condition (25) holding with equality we can rewrite condition (22) as

$$P \leq \frac{c_n \sum_{i=1}^n \frac{c_n - c_i}{g_i} + \sum_{i=l}^n \left(\frac{c_i + c_l}{2}\right)\left(\frac{c_i - c_l}{2gi}\right)}{\sum_{i=1}^n \frac{c_n - c_i}{g_i} + \sum_{i=l}^n \frac{c_i - c_l}{2gi}}. \tag{65}$$

From (25) holding with equality, it follows that $B \geq 0$ is equivalent to $P \leq c_n$, which is redundant with (65).

Similarly, from condition (25) holding with equality we can rewrite condition (23) as

$$P \geq \frac{c_n \sum_{i=1}^n \frac{c_n - c_i}{g_i} + \sum_{i=l}^n \left(\frac{c_i + c_{l-1}}{2}\right)\left(\frac{c_i - c_{l-1}}{2gi}\right)}{\sum_{i=1}^n \frac{c_n - c_i}{g_i} + \sum_{i=l}^n \frac{c_i - c_{l-1}}{2gi}}, \tag{66}$$

which implies $P \geq 0$.

Additionally, consider any instance of problem (CAP) such that $P(Q^{\mathrm{U}}) = P(Q^*) = P$ and condition (25) holds with equality, that is $q_n^{\mathrm{U}} = 0$. From equations (20) and (21) for $u = n$, together with the expression for B from (25) with equality, it follows that the inequality (27) simplifies to

$$\sum_{i=1}^n \frac{2c_n - P - c_i}{g_i} + \sum_{i=l}^n \frac{P - c_i}{2g_i} - \sqrt{\left(\sum_{i=l}^n \frac{P - c_i}{2g_i}\right)^2 + \sum_{i=l}^n \frac{c_n - P}{g_i} \sum_{i=1}^n \frac{c_n - c_i}{g_i} + \sum_{i=l}^n \frac{1}{g_i} \sum_{i=l}^n \left(\frac{c_i - c_l}{2}\right)^2 \frac{1}{g_i} - \left(\sum_{i=l}^n \frac{c_i - c_l}{2g_i}\right)^2} \geq 0. \tag{67}$$

Hence, the proof reduces to showing that any minimum of the left hand side of equation (67) must be attained when at least one of (65) or (66), equivalently one of (22) or (23), also holds with equality.

Denote by $f(P)$ the function defined by the left hand side of inequality (67), and note that $f(P)$ is concave. Moreover, the simplified expressions (65) and (66) define a non-empty compact interval for P. Therefore, it follows that $f(P)$ must attain its minimum at one of the extremes of the feasible interval. Equivalently, when at least one of the conditions (22) or (23) also holds with equality.

Proof of Proposition 4 .

Proof. For any number of firms in the market $n \geq 2$, and for any parameter $c > 0$, consider the instance of problem (CAP) defined by the budget $B = \frac{c}{(n-1)}\left(\sqrt{4 + \frac{n-1}{4}} - 2\right) > 0$, and the inverse demand function $P(Q) = a - bQ$. Where $b = \frac{\epsilon 4\left(\sqrt{4+\frac{n-1}{4}} - 2\right)}{n\left(\frac{\sqrt{n-1}}{2} - \left(\sqrt{4+\frac{n-1}{4}} - 2\right)\right)^2 - \frac{4c}{(c-\epsilon)}\left(\sqrt{4+\frac{n-1}{4}} - 2\right)} > 0$ and $a = c - \frac{(c-\epsilon)}{c}B + \frac{c}{(c-\epsilon)}b > 0$, for $\epsilon > 0$ small enough.

The remainder of the proof consists of guessing and checking that $q_i^{\mathrm{U}} = 0$ for each $i \in \{2, \ldots, n\}$, and $q_1^{\mathrm{U}} = Q^{\mathrm{U}} = \frac{c}{(c-\epsilon)}$, is the market equilibrium induced by uniform co-payments. As well as $q_1^* = \frac{c}{(c-\epsilon)} + \frac{2 - \sqrt{4+\frac{n-1}{4}} + \frac{\sqrt{n-1}}{2}}{2(n-1)} - \left(\frac{c}{c-\epsilon}\right)\frac{2\left(\sqrt{4+\frac{n-1}{4}} - 2\right)}{(n-1)\left(2 - \sqrt{4+\frac{n-1}{4}} + \frac{\sqrt{n-1}}{2}\right)}$, $q_i^* = \frac{2 - \sqrt{4+\frac{n-1}{4}} + \frac{\sqrt{n-1}}{2}}{2(n-1)}$ for each $i \in \{2, \ldots, n\}$, and $Q^* =$

$\frac{c}{(c-\epsilon)} + \frac{n\left(2-\sqrt{4+\frac{n-1}{4}}+\frac{\sqrt{n-1}}{2}\right)}{2(n-1)} - \left(\frac{c}{c-\epsilon}\right)\frac{2\left(\sqrt{4+\frac{n-1}{4}}-2\right)}{(n-1)\left(2-\sqrt{4+\frac{n-1}{4}}+\frac{\sqrt{n-1}}{2}\right)}$, is the market equilibrium induced by optimal co-payments, and it is omitted.

We conclude by evaluating the ratio of

$$\frac{Q^*}{Q^{\mathrm{U}}} = 1 + \left(\frac{c-\epsilon}{c}\right)\frac{n\left(2-\sqrt{4+\frac{n-1}{4}}+\frac{\sqrt{n-1}}{2}\right)}{2(n-1)} - \frac{2\left(\sqrt{4+\frac{n-1}{4}}-2\right)}{(n-1)\left(2-\sqrt{4+\frac{n-1}{4}}+\frac{\sqrt{n-1}}{2}\right)} \xrightarrow{n\to\infty} 1 + \frac{(c-\epsilon)}{c} \xrightarrow{\epsilon\to 0} 2.$$

References

A2S2. 2014. Who/ema prequalified act manufacturers. URL `http://www.a2s2.org/market-data/map-of-stakeholders/\#1`.

AMFm Independent Evaluation Team. 2012. Independent evaluation of phase 1 of the affordable medicines facility - malaria (amfm), multi-country independent evaluation report: Final report. Tech. rep., The Global Fund. URL `www.theglobalfund.org/documents/amfm/AMFm_2012IEPhase1FullFinal_Report_en/`.

Arrow, K. J., C. B. Panosian, H. Gelband. 2004. *Saving Lives, Buying Time: Economics of Malaria Drugs in an Age of Resistance*. National Academies Press, Washington, DC.

Aydin, G., E. L. Porteus. 2009. Manufacturer-to-retailer versus manufacturer-to-consumer rebates in a supply chain. *Retail Supply Chain Management*. Springer, 237–270.

Bernstein, F., A. Federgruen. 2005. Decentralized supply chains with competing retailers under demand uncertainty. *Management Science* **51**(1) 18–29.

Bertsekas, D. P. 1999. *Nonlinear programming*. Athena Scientific, Belmont, MA.

Bertsimas, D., J. N. Tsitsiklis. 1997. *Introduction to linear optimization*. Athena Scientific Belmont, MA.

Boulton, I. 2011. Background paper: Forum for africa-based pharmaceutical manufacturers and potential investors. Tech. rep., The Global Fund. URL `http://www.theglobalfund.org/documents/amfm/AMFm_ForumForAfricaBasedPharmaceuticalManufacturersAndPotentialInvestors_Background_Paper_en/`.

Brito, D. L., E. Sheshinski, M. D. Intriligator. 1991. Externalities and compulsary vaccinations. *Journal of Public Economics* **45**(1) 69 – 90.

Cachon, G. P. 2003. Supply chain coordination with contracts. A. G. de Kok, S. C. Graves, eds., *Handbooks in Operations Research and Management Science, Vol 11: Supply Chain Management*. Elsevier, Amsterdam, North - Holland, 229 – 339.

Charnes, A., W. W. Cooper. 1962. Programming with linear fractional functionals. *Naval Research Logistics Quarterly* **9**(3-4) 181196.

Cohen, M., R. Lobel, G. Perakis. 2012. Consumer subsidies with industry response for green technology adoption. *Under review* .

Corbett, C. J., U. S. Karmarkar. 2001. Competition and structure in serial supply chains with deterministic demand. *Management Science* **47**(7) 966–978.

Deo, S., C. J. Corbett. 2009. Cournot competition under yield uncertainty: The case of the u.s. influenza vaccine market. *Manufacturing and Service Operations Management* **11**(4) 563 – 576.

Dong, L., P. Kouvelis, P. Su. 2013. Global facility network design in the presence of competition. *European Journal of Operational Research* **228**(2) 437–446.

Farahat, A., G. Perakis. 2011. Technical note-a comparison of bertrand and cournot profits in oligopolies with differentiated products. *Operations research* **59**(2) 507–513.

Fullerton, D., G. E. Metcalf. 2002. Chapter 26 tax incidence. Alan J. Auerbach, Martin Feldstein, eds., *Handbook of Public Economics*, vol. 4. Elsevier, 1787 – 1872. doi:10.1016/S1573-4420(02)80005-2. URL `http://www.sciencedirect.com/science/article/pii/S1573442002800052`.

Krass, D., T. Nedorezov, A. Ovchinnikov. 2013. Environmental taxes and the choice of green technology. *Production and Operations Management* **22**(5) 1035–1055.

Lobel, R., G. Perakis. 2012. Consumer choice model for forecasting demand and designing incentives for solar technology. *Under review* .

Marcotte, P., M. Patriksson. 2006. Traffic equilibrium. C. Barnhart, G. Laporte, eds., *Handbooks in Operations Research and Management Science, Vol. 14: Transportation*. Elsevier, Amsterdam, North - Holland, 623 – 713.

Roughgarden, T. 2005. *Selfish Routing and the Price of Anarchy*. MIT Press, Cambridge, MA.

Roughgarden, T., É. Tardos. 2002. How bad is selfish routing? *Journal of the ACM (JACM)* **49**(2) 236–259.

Snyder, C. M., W. Begor, E. R. Berndt. 2011. Economic perspectives on the advance market commitment for pneumococcal vaccines. *Health Affairs* **30**(8) 1508 – 1517.

Stackelberg, H. V. 1952. *The Theory of Market Economy*. Oxford University Press, London.

Taylor, T. A., W. Xiao. 2014. Subsidizing the distribution channel: Donor funding to improve the availability of malaria drugs. *Management Science,* Forthcoming.

The authors. 2014. The effectiveness of uniform subsidies in increasing market consumption. *Under review* .

Tirole, Jean. 1988. *The theory of industrial organization*. MIT press.

White, N. J. 2008. Qinghaosu (artemisinin): the price of success. *Science* **320** 330 – 334.

World Health Organization. 2013. World malaria report: 2013. Tech. rep., WHO. URL `http://www.who.int/malaria/publications/world_malaria_report_2013/report/en/index.html`.

Appendix 2: Online Appendix

2.1. Proofs of Section 5

We restate here a characterization of any optimal solution to problem (UBP) in Section 5.

LEMMA 5 **(The authors (2014))**. *Assume that the marginal cost functions $h_i(q_i)$ are non-negative, increasing, and differentiable in $[0,\bar{Q})$; and that the inverse demand function $P(Q)$ is non-negative, decreasing, and differentiable in $[0,\bar{Q}]$. Then, any optimal solution to the upper bound problem (UBP) must satisfy the budget constraint (32) with equality, and also satisfy the following condition:*

$$\text{If } \; q_i > 0, \; \text{ then } \; (h_i(q_i)q_i)' \leq (h_j(q_j)q_j)', \; \text{ for each } i,j \in \{1,\dots,n\}. \tag{68}$$

Proof of Proposition 5

Proof. For any number of firms in the market $n \geq 2$, consider the instance of problem (UBP) defined by the parameter $c = \sqrt{n}(n-\epsilon-1)(3+\sqrt{n-\epsilon}) > 0$, for $\epsilon > 0$ small enough, the budget $B = \epsilon$, and the inverse demand function $P(Q) = a - bQ$, where $b = \epsilon\left(\frac{3+\sqrt{n-\epsilon}}{\sqrt{n}} - \frac{2}{1+\sqrt{n-\epsilon}}\right) > 0$, and $a = c + b - \epsilon > 0$. The remainder of the proof consists of guessing and checking that $q_i^{\mathrm{U}} = 0$ for each $i \in \{2,\dots,n\}$, and $q_1^{\mathrm{U}} = Q^{\mathrm{U}} = 1$, is the market equilibrium induced by uniform co-payments. As well as $q_1^{\mathrm{UB}} = \frac{n+\sqrt{n-\epsilon}}{2n} > 0$, $q_i^{\mathrm{UB}} = \frac{\sqrt{n-\epsilon}}{2n} > 0$ for each $i \in \{2,\dots,n\}$, and $Q^{\mathrm{UB}} = \frac{1+\sqrt{n-\epsilon}}{2} > 0$, is the market equilibrium induced by the optimal policy of problem (UBP), and it is omitted.

Proof of Proposition 6

Proof. For any $n \geq 2$, and functions $h(q_i)$ such that $h(q_i)q_i$ is convex for each firm $i \in \{1,\dots,n\}$, $P(Q)$ where $\sum_{i=1}^n q_i = Q$, and budget $B \geq 0$, let $(\mathbf{q}^{\mathrm{U}}, Q^{\mathrm{U}})$ and $(\mathbf{q}^{\mathrm{UB}}, Q^{\mathrm{UB}})$ be the market outputs induced by uniform co-payments and by the optimal policy for problem (UBP), respectively.

On the other hand, for any $Q \geq 0$, let us define $\hat{\mathbf{q}}(Q)$ as the minimizer of the following auxiliary problem,

$$\min_{\mathbf{q}} \sum_{i=1}^n h_i(q_i)q_i \quad \text{s.t.} \quad \sum_{i=1}^n q_i = Q, \quad q_i \geq 0 \text{ for each } i \in \{1,\dots,n\}.$$

From $h(q_i)q_i$ convex it follows that $\hat{\mathbf{q}}(Q)$ is well defined. Moreover, the optimal objective value is increasing in the parameter Q. We show that if we assume that the marginal cost functions $h(q_i)$ are additionally concave then we have that

$$\sum_{i=1}^n h_i(q_i^{\mathrm{U}})q_i^{\mathrm{U}} \leq \frac{2}{3}\sum_{i=1}^n h_i(\hat{q}_i(2Q^{\mathrm{U}}))\hat{q}_i(2Q^{\mathrm{U}}). \tag{69}$$

In the same spirit as Roughgarden and Tardos (2002), let us define the function $\bar{h}_i(q_i) = \begin{cases} h_i(q_i^{\mathrm{U}}) & \text{if } q_i \leq q_i^{\mathrm{U}} \\ h_i(q_i) & \text{if } q_i > q_i^{\mathrm{U}} \end{cases}$ Then,

$$\sum_{i=1}^n \bar{h}_i(\hat{q}_i(2Q^{\mathrm{U}}))\hat{q}_i(2Q^{\mathrm{U}}) - \sum_{i=1}^n h_i(\hat{q}_i(2Q^{\mathrm{U}}))\hat{q}_i(2Q^{\mathrm{U}}) \leq \frac{1}{2}\sum_{i=1}^n h_i(q_i^{\mathrm{U}})q_i^{\mathrm{U}}. \tag{70}$$

Where the inequality is valid for any concave increasing functions $h_i(q_i)$, and tight only if all the functions are linear. Additionally, note that we also have that

$$\sum_{i=1}^{n} \bar{h}_i(\hat{q}_i(2Q^{\mathrm{U}}))\hat{q}_i(2Q^{\mathrm{U}}) \geq \sum_{i=1}^{n} h_i(q_i^{\mathrm{U}})\hat{q}_i(2Q^{\mathrm{U}}) \geq 2\sum_{i=1}^{n} h_i(q_i^{\mathrm{U}})q_i^{\mathrm{U}}. \tag{71}$$

Where the first inequality follows from the definition of the function $\bar{h}_i(q_i)$, and the second inequality follows from the market equilibrium condition (6) together with the fact that all the active firms get the same co-payment y^{U} under uniform co-payments. Combining equations (70) and (71) we obtain equation (69). We conclude by noticing that

$$\begin{aligned} P(Q^{\mathrm{U}})Q^{\mathrm{U}} + B &= \sum_{i=1}^{n} h_i(q_i^{\mathrm{U}})q_i^{\mathrm{U}} \leq \frac{2}{3}\sum_{i=1}^{n} h_i(\hat{q}_i(2Q^{\mathrm{U}}))\hat{q}_i(2Q^{\mathrm{U}}) \\ &\leq \frac{2}{3}\sum_{i=1}^{n} h_i(\hat{q}_i(Q^{\mathrm{UB}}))\hat{q}_i(2Q^{\mathrm{UB}}) = \frac{2}{3}\left(P(Q^{\mathrm{UB}})Q^{\mathrm{UB}} + B\right). \end{aligned} \tag{72}$$

Where the first equality follows from the the budget being exhausted by uniform subsidies, the first inequality follows from equation (69), the second inequality follows from the assumption $2Q^{\mathrm{U}} \leq Q^{\mathrm{UB}}$, and the last equality follows because from Lemma 5, $(\mathbf{q}^{\mathrm{UB}}, Q^{\mathrm{UB}})$ also exhausts the budget. Inequality (72) implies equation (39). The proof of equation (38) follows from Theorem 3.1 in Roughgarden and Tardos (2002) specialized to our setting, and it is omitted.

LEMMA 6. *For any $n \geq 2$, $P(0) > 0$ and n-dimensional vectors $\boldsymbol{c}$, $\boldsymbol{g}$ such that $0 \leq c_1 \leq \ldots \leq c_n \leq P(0)$ and $0 < g_1 \leq \ldots \leq g_n$, it must be the case that*

$$(n-1)\left(\sum_{i=1}^{n} \frac{c_n - c_i}{g_i}\right)^2 + \left(\sum_{i=1}^{n} \frac{c_i}{g_i}\right)^2 - \sum_{i=1}^{n} \frac{c_i^2}{g_i} \sum_{i=1}^{n} \frac{1}{g_i} \geq 0. \tag{73}$$

Proof. For any $n \geq 2$, $P(0) > 0$ and n-dimensional vector $\mathbf{g}$ such that $0 < g_1 \leq \ldots \leq g_n$, the left hand side of the inequality (73) is a continuous function of the n-dimensional vector $\mathbf{c}$. It follows that it must attain a minimum in the compact feasible set defined by $0 \leq c_1 \leq \ldots \leq c_n \leq P(0)$.

We show that any minimum of the left hand side of inequality (73) over $0 \leq c_1 \leq \ldots \leq c_n \leq P(0)$ must be such that each c_i is either equal to 0 or to c_n. First, assume for a contradiction that there exists a minimum such that at least two indexes have values strictly between 0 and c_n. Let i be the first index such that $c_i > 0$, and j be the last index such that $c_j < c_n$, then it follows that we can increase c_j and decrease c_i by the same $\epsilon > 0$ small enough, and obtain a strictly smaller value of the left hand side of inequality (73), a contradiction. Hence, there can be at most one index $i \in \{1\ldots, n\}$ such that $0 < c_i < c_n$. However, note that then the derivative of the left hand side of inequality (73) with respect to c_i is $\frac{2}{g_i}\left(c_n\left(\sum_{j=1}^{n}\frac{1}{g_j} - \sum_{j=1}^{i}\frac{n}{g_j}\right) + c_i\left(\frac{n}{g_i} - \sum_{j=1}^{n}\frac{1}{g_j}\right)\right) < 0$. Where the inequality follows from $0 < g_1 \leq \ldots \leq g_n$ and $c_i < c_n$. Therefore, we can increase c_i and obtain a strictly smaller value of the left hand side of inequality (73), a contradiction.

2.2. Proofs of Section 6

Proof of Proposition 8

Proof. For any $n \geq 2$ there is a finite set of possible combinations of indexes $l, m, u \in \{1, \ldots, n\}$, $l \leq m$. For each combination, Q^{U}/Q^* has a closed form given by (42) and (43), continuous on the problem parameters. Moreover, from $0 \leq c_1 \leq \ldots \leq c_n \leq a$, $B \geq$, and conditions (45)-(48) it follows that the feasible set of problem parameters is compact. Hence, there exists an instance of problem (CAP) that minimizes Q^{U}/Q^*.

Consider any instance of problem (CAP) $a, b, \mathbf{c}$, and B that minimizes the ratio Q^{U}/Q^*. From Lemma 7 it follows that we can assume $c_1 > 0$. Let $(\mathbf{q}^{\mathrm{U}}, Q^{\mathrm{U}})$ be the solution induced by uniform co-payments. Assume for a contradiction that $q_n^{\mathrm{U}} > 0$. Then, we can increase the value of c_n and reduce the value of c_1 by the same $\epsilon > 0$ sufficiently small, without affecting Q^{U}, while obtaining a strictly larger value for Q^*. Specifically, let $(\mathbf{q}^*, Q^*)$ be an optimal solution to the original instance of problem (CAP). Consider the modified solution $(\hat{\mathbf{q}}, Q^* + \gamma)$, where $\hat{q}_1 = q_1^* + \delta + \gamma$, $\hat{q}_i = q_i^*$ for each $i \in \{2, \ldots, n-1\}$, and $\hat{q}_n = q_n^* - \delta$, for $\delta > 0$ and $\gamma > 0$ such that $\epsilon = b(\delta + 2\gamma)$, where δ is close enough to $\frac{\epsilon}{b} > 0$, and γ is arbitrarily smaller than δ. This solution is feasible for the modified instance of problem (CAP), and attains an objective value strictly larger than Q^*.

LEMMA 7. *For any given number $n \geq 2$ of Cournot competitors with constant marginal costs $c_i \geq 0$, for each firm $i \in \{1, \ldots, n\}$, facing a linear inverse demand function $P(Q) = a - bQ$, where $a > 0$, $b > 0$. Any instance of the co-payments allocation problem problem (CAP) with $c_1 \geq 0$, and for any scaling parameter $\delta \geq 0$, there exists a modified instance with $c_1 = \delta$ such that the modified instance has the same set of optimal solutions, which attain the same objective value.*

Proof. Consider the modified instance $\hat{a} = (a + \delta - c_1)$, $\hat{\mathbf{c}} = (\mathbf{c} + (\delta - c_1)\mathbf{e})$, where $\mathbf{e}$ is a vector of ones. Any feasible solution in the original instance is feasible in the modified instance, and it attains the same objective value, and viceversa.

PROPOSITION 13. *For any given number of firms in the market $n \geq 2$, and demand parameters $a > 0$, $b > 0$, any optimal solution to problem (WCP_1) must be such that condition (46) is binding.*

Proof. From Lemma 4 for $l = 1$

$$
\begin{aligned}
WC(1,n) \equiv \quad & \min_{B,\mathbf{c}} \quad \frac{Q^{\mathrm{U}}(\mathbf{c})}{Q_1^*(B,\mathbf{c})} = \frac{2(n+1)\left(nc_n - \sum_{i=1}^n c_i\right)}{na - \sum_{i=1}^n c_i + \sqrt{*_1}(B,\mathbf{c})} \\
& \text{s.}t. \quad 0 \leq c_1 && (74) \\
& \qquad\quad c_i \leq c_{i+1}, \text{ for each } i \in \{1, \ldots, n-1\} && (75) \\
(WCP_1) \; & \qquad\quad c_n \leq a && (76) \\
& \qquad\quad bB = \left(nc_n - \sum_{i=1}^n c_i\right)\left((n+1)c_n - \sum_{i=1}^n c_i - a\right) && (77)
\end{aligned}
$$

$$bB \geq \frac{1}{4(n+1)} \left(\sum_{i=2}^{n} (c_i - c_1) \right) \left(2a + \sum_{i=2}^{n} c_i - (n+1)c_1 \right) - \sum_{i=2}^{n} \frac{(c_i - c_1)^2}{4} \quad (78)$$

where constraints (77) and (78) correspond to (48) and (45) in problem (WCP_l). From $l = 1$ we can drop (46) from (WCP_l). Moreover, we drop constraints (54) and (47) from (WCP_l) as well, because $B \geq 0$ is redundant with constraints (77) and (78), and (47) is redundant with (45).

Let $(B^*, \mathbf{c}^*)$ be an optimal solution to problem (WCP_1). Note that if the k largest variables c_i^* are equal to c_n^*, with $k \in \{1, \ldots, n-1\}$, then the objective function is strictly increasing in c_n. It follows that c_n^* must attain its lower bound, otherwise we could strictly improve the objective by decreasing it. Then, from $k = 1$ it follows that either constraint (78) is tight, or we must have $c_n^* = c_{n-1}^*$. If constraint (78) is tight, we are done. Therefore, assume that $c_n^* = c_{n-1}^*$. In fact, by iterating this argument for each $k \in \{2, \ldots, n-2\}$, and conclude that either constraint (78) is tight, or we must have $c_n^* = c_i^*$ for each $i \in \{2, \ldots, n\}$. Again, if constraint (78) is tight, we are done. Therefore, assume that $c_n^* = c_i^*$ for each $i \in \{2, \ldots, n\}$. It follows that constraint (78) simplifies to $c_n^* \geq \frac{(3n+1)}{(5n+3)} a > 0$. Finally, from $k = n-1$ it follows that c_n^* must attain its lower bound, hence constraint (78) must be tight. This concludes the proof.

Proposition 14. *For any given number of firms in the market $n \geq 2$, demand parameters $a > 0$, $b > 0$, and for each $l \in \{2, \ldots, n\}$, it must be the case that $WC(l,n) \geq \frac{2nl - 2n + 2l - 2}{2nl - n + l - 1}$.*

Proof. The proof structure is the following. We will consider a mathematical programming relaxation of problem (WCP_l), denoted by (LBP_l), whose optimal solution provides a lower bound on $WC(l,n)$, for any $n \geq 2$, and for each $l \in \{2, \ldots, n\}$. We will reformulate this relaxation as a linear program, and we will use strong duality to obtain its optimal objective value in closed form.

First note that for any given number of firms in the market $n \geq 3$, and demand parameter $a > 0$, problem (LBP_l) below is a mathematical programming relaxation of problem (WCP_l).

$$(LBP_l) \quad \begin{aligned} \min_{\mathbf{c}} \ & \frac{2(n+1)\left(nc_n - \sum_{i=1}^{n} c_i\right)}{2na - 2\sum_{i=1}^{l-2} c_i - (n-l+3)c_{l-1} - \sum_{i=l}^{n} c_i} \\ \text{s.}t. \ & 0 \leq c_1 && (79) \\ & c_i \leq c_{i+1}, \text{ for each } i \in \{1, \ldots, n-1\} && (80) \\ & c_n \leq a && (81) \\ & (n+1)c_n - \sum_{i=1}^{n} c_i - a \geq 0. && (82) \end{aligned}$$

The relaxation (LBP_l) is a linear fractional program. Hence, from Charnes and Cooper (1962), it follows that the relaxation (LBP_l) is equivalent to the following linear program

$$\min_{t,\mathbf{x}} \ nx_n - \sum_{i=1}^{n} x_i$$

$$
(LP_l) \quad \begin{aligned}
\text{s.}t. \quad & 0 \leq x_2 && (83) \\
& x_i \leq x_{i+1} \text{ for each } i \in \{2,\dots,n-1\} && (84) \\
& x_n \leq at && (85) \\
& (n+1)x_n - \sum_{i=1}^{n} x_i - at \geq 0. && (86) \\
& 2nat - 2\sum_{i=1}^{l-2} x_i - (n-l+3)x_{l-1} - \sum_{i=l}^{n} x_i = 1 && (87) \\
& t \geq 0. && (88)
\end{aligned}
$$

Note that $x_i = 0$ for each $i \in \{1,\dots,l-1\}$, $x_i = \frac{1}{2nl-n+l-1}$ for each $i \in \{l,\dots,n\}$, $t = \frac{l}{(2nl-n+l-1)a}$ is a primal feasible solution. On the other hand, $\lambda = \frac{(l-1)}{2nl-n+l-1}$, $\gamma = -2n\lambda$, $u_i = (l-i-1)\gamma - (n+l-2i-1)\lambda + l - i - 1$ for each $i \in \{2,\dots,l-2\}$, $u_{l-1} = 0$, $u_i = -(i+1)\gamma - (n-i)\lambda - i$ for each $i \in \{l,\dots,n-1\}$, and $u_n = 0$, is a dual feasible solution. Moreover, both attain the same objective value $\frac{(l-1)}{2nl-n+l-1}$. Hence, from strong duality in linear programming, it follows that they are primal and dual optimal, respectively, see for example Bertsimas and Tsitsiklis (1997). Therefore, the associated solution $c_i = 0$ for each $i \in \{1,\dots,l-1\}$, $c_i = \frac{a}{l}$ for each $i \in \{l,\dots,n\}$, is optimal for problem (LBP_l), and we conclude $WC(l,n) \geq \frac{2nl-2n+2l-2}{2nl-n+l-1}$ for any $n \geq 2$, $a > 0$, $b > 0$, and for each $l \in \{2,\dots,n\}$.

Proposition 15. *For any given number of firms in the market $n \geq 3$, and demand parameters $a > 0$, $b > 0$, it must be the case that $WC(2,n) \leq WC(3,n)$.*

Proof. For any given $n \geq 3$, $a > 0$, $b > 0$, consider any optimal solution $(B^*, \mathbf{c}^*)$ to problem (WCP_3). Note that if (46) is tight, then it follows that $WC(2,n) \leq WC(3,n)$ and we are done. Similarly, if (45) is tight, then it follows that $WC(2,n) \leq WC(4,n) \leq WC(3,n)$, where the first inequality follows from the case $l = 4$ in Proposition 14, and we are done in this case as well. Hence, without loss of generality we will assume that conditions (45) and (46) are *loose* for $(B^*, \mathbf{c}^*)$.

Lemma 8 below shows that then $(B^*, \mathbf{c}^*)$ must be such that $c_2^* = c_1^*$, and $c_i^* = c_n^*$, for each $i \in \{3,\dots,n\}$. Therefore, without loss of generality we focus on solutions with this structure. Moreover, from Lemma 7 we will assume, without loss of generality, that $c_1^* = 0$. It follows that problem (WCP_3) simplifies to the following one variable optimization problem.

$$
\begin{aligned}
\min_{c_n} \quad & \frac{Q_3^{\mathrm{U}}(c_n)}{Q_3^*(c_n)} = \frac{12(n+1)c_n}{(5n+2)a - 3(n-2)c_n + ((n-2)(3c_n - a)(9(3n+2)c_n - (n-2)a))^{1/2}} \\
\text{s.}t. \quad & \frac{a}{3} \leq c_n && (89) \\
& c_n \leq \frac{2(5n+2)a}{9(3n+2)}, && (90)
\end{aligned}
$$

where constraint (89) is equivalent to condition (46), and constraint (90) is equivalent to both conditions (45) and (47).

Now we show that for any given $n \geq 3$, $a > 0$, any optimal solution c_n^* to problem $(SWCP_3)$ must have an objective value of at least $WC(2,n)$. Recall that if at any optimal solution to problem $(SWCP_3)$ constraints (89) or (90) are tight then we are done. It follows that, without loss of generality, we can focus on values of c_n such that $\frac{d\left(Q_3^{\mathrm{U}}(c_n^*)/Q_3^*(c_n^*)\right)}{dc_n} = 0$. After simplifying, this condition is equivalent to

$$((n-2)(3c_n^* - a)(9(3n+2)c_n^* - (n-2)a))^{1/2} = 3(n-2)c_n^* - \frac{(n-2)^2}{5n+2}a. \tag{91}$$

By substituting expression (91) into the objective function, it follows that any interior stationary point c_n^* must be such that its objective value has the following simplified expression $\frac{Q_3^{\mathrm{U}}(c_n^*)}{Q_3^*(c_n^*)} = \frac{(5n+2)c_n^*}{2na}$. Furthermore, equation (91) is quadratic in c_n^* and its unique non-negative solution is $c_n^* = \left(\frac{3n+\sqrt{6n(n+1)}}{3(5n+2)}\right) a$. Hence, it follows that any interior stationary point c_n^* attains an objective value of $\frac{3+\sqrt{6+6/n}}{6} \geq \frac{2+\sqrt{2+2/n}}{4} \geq WC(2,n)$, where the first inequality holds for any $n \geq 1$, and the second inequality follows from the fact that the left hand side is attained by the candidate instance for the case $l = 2$ from Proposition 9.

LEMMA 8. *For any given number of firms in the market $n \geq 3$, and demand parameters $a > 0$, $b > 0$, any optimal solution $(B^*, \mathbf{c}^*)$ to problem (WCP_3) for which conditions (45) and (46) are loose must be such that $c_2^* = c_1^*$, and $c_i^* = c_n^*$, for each $i \in \{3, \dots, n\}$.*

Proof. For any given $n \geq 3$, $a > 0$, $b > 0$, consider any optimal solution $(B^*, \mathbf{c}^*)$ to problem (WCP_3) such that conditions (45) and (46) are loose. Note that, for each index $i \in \{2, \dots, n-1\}$, it must be the case that either $c_i^* = c_1^*$ or $c_i^* = c_n^*$. Assume for a contradiction that $i \in \{2, \dots, n-1\}$ is the largest index such that $c_1^* < c_i^* < c_n^*$. Recall that from Lemma 7 it follows that we can assume without loss of generality that $c_1^* = \delta > 0$. Then, note that we can transfer an arbitrarily small $\epsilon > 0$ from c_1^* to c_i^* and strictly improve this solution, while maintaining feasibility for problem (WCP_3), a contradiction. To conclude, note that assuming $l = 3$ implies $c_2^* < c_3^*$, otherwise if $c_2^* = c_3^*$ then firm 2 would get a co-payment whenever firm 3 does, contradicting the definition of the index l in Proposition 1. It follows that $(B^*, \mathbf{c}^*)$ must have the structure given in the statement of the proposition.

PROPOSITION 16. *For any given number of firms in the market $n \geq 2$, demand parameter $a > 0$, and budget B, problem $(RWCP_{2,1})$ below is a mathematical programming relaxation of problem (WCP_2), whose optimal objective value provides a lower bound on $WC(2,n)$*

$$\min_{c_n} \quad \frac{4(n+1)c_n}{(3n+1)a - 2(n-1)c_n + \sqrt{(n-1)(a-2c_n)\left((n-1)a - 2(5n+3)c_n\right)}}$$

$$s.t. \quad \frac{a}{2} \le c_n \tag{92}$$

$$(RWCP_{2,1}) \; c_n \le a. \tag{93}$$

Proof. First, note that for any $n \ge 2$, demand parameter $a > 0$, and budget B, problem $(RWCP_2)$ below is a mathematical programming relaxation of problem (WCP_2).

$$\min_{B,\mathbf{c}} \frac{Q^{\mathrm{U}}(\mathbf{c})}{Q_2^*(B,\mathbf{c})}$$

$$\text{s.}t. \quad o \le c_1 \tag{94}$$

$$c_i \le c_{i+1}, \text{ for each } i \in \{1,\dots,n-1\} \tag{95}$$

$$(RWCP_2) \qquad c_n \le a \tag{96}$$

$$(n+1)c_n - \sum_{i=1}^{n} c_i - a \ge 0. \tag{97}$$

Now we show that, without loss of generality, solving problem $(RWCP_2)$ is equivalent to solving one of the following one variable optimization problems, for some index $k \in \{1,\dots,n-1\}$.

$$\min_{c_n} \frac{Q_k^{\mathrm{U}}(c_n)}{Q_{2,k}^*(c_n)} = \frac{4(n+1)kc_n}{(3n+1)a - 2(n-k)c_n + \sqrt{*_{2,k}}(c_n)}$$

$$(RWCP_{2,k}) \quad \text{s.}t. \quad \frac{a}{k+1} \le c_n \tag{98}$$

$$c_n \le a. \tag{99}$$

Note that for any number of firms in the market $n \ge 2$, and demand parameters $a > 0$, $b > 0$, any optimal solution $\mathbf{c}^*$ to problem $(RWCP_2)$ must satisfy that there exists an index $k \in \{1,\dots,n-1\}$ such that $c_i^* = c_1^*$, for each $i \in \{1,\dots,k\}$, and $c_i^* = c_n^*$, for each $i \in \{k+1,\dots,n\}$. The proof of this statement is identical to the first part of the proof of Lemma 8, and it is omitted. It follows that, without loss of generality, we can focus on solutions to problem $(RWCP_2)$ with a special structure, which can be parametrized by the number of firms k with their marginal cost equal to c_1^*. Moreover, from Lemma 7, we assume that $c_1^* = 0$. Then, in this case, the function $\sqrt{*_l}(B,\mathbf{c})$ in equation (50) simplifies to

$$\sqrt{*_{2,k}}(c_n) \equiv \left((n-1)\left((n-1)a^2 + 2(k+1)(4nk+n+3k)c_n^2 - 4(2nk+n+k)ac_n\right)\right)^{1/2}.$$

Similarly, Q^* and and Q^{U}simplify to $Q_{2,k}^*(c_n) \equiv \frac{(3n+1)a - 2(n-k)c_n + \sqrt{*_{2,k}}(c_n)}{4(n+1)b}$, and $Q_k^{\mathrm{U}}(c_n) \equiv \dfrac{kc_n}{b}$.

Finally, we show that for any $n \ge 3$, $a > 0$, $b > 0$, and for any index $k \in \{2,\dots,n-1\}$, there is no feasible solution to problem $(RWCP_{2,k})$ that attains an objective value smaller than $\frac{2+\sqrt{2+2/n}}{4}$. The conclusion then follows from the observation that this lower bound is attained by the candidate instance from Proposition 9, which is feasible for problem $(RWCP_{2,1})$

Note that the objective function in problem $(RWCP_{2,k})$ is quasiconvex. Hence, its minimum must be attained either at one of the extremes of the feasible interval $c_n \in \left[\frac{a}{k+1}, a\right]$, or at an interior stationary point. We will analyze each one of these cases, and show that none of them attains an objective value smaller than $\frac{2+\sqrt{2+2/n}}{4}$.

(i) If $c_n = \frac{a}{k+1}$, then the objective function of problem $(RWCP_{2,k})$ evaluates to

$$\frac{4(n+1)k}{(n+1)(3k+1)+\sqrt{(n-1)(n+1)(k-1)(k+1)}} \geq \frac{4k}{3k+1+\sqrt{k^2-1}} \geq \frac{8}{7+\sqrt{3}} \geq \frac{2+\sqrt{2+2/n}}{4}.$$

Where the first inequality follows from the left hand side decreasing in n, and taking the limit as $n \to \infty$. The second inequality follows from the left hand side increasing in k, for any $k \in \{2,\ldots,n-1\} \geq \sqrt{2}$, $n \geq 3$, and taking $k = 2$. Finally, the last inequality holds for any $n \geq 3$.

(ii) If $c_n = a$, then the objective function of problem $(RWCP_{2,k})$ evaluates to

$$\frac{4(n+1)k}{n+2k+1+\sqrt{(n-1)\left((8k^2+2k-1)n+6k^2+2k-1\right)}} \geq \frac{4k}{1+\sqrt{8k^2+2k-1}} \geq \frac{4k}{3k+2} \geq \frac{2+\sqrt{2+2/n}}{4}.$$

Where the first inequality follows from the left hand side being decreasing in n, and taking the limit as $n \to \infty$. The second follows from $(3k+1)^2 \geq (8k^2+2k-1)$, for any $k \in \{2,\ldots,n-1\}$, $n \geq 3$. The third inequality follows from the left hand side being increasing in k, and taking $k = 2$. It holds for any $n \geq 1$.

(iii) Any interior stationary solution to problem $(RWCP_{2,k})$ must satisfy $\frac{dQ_k^{\mathrm{U}}(c_n^*)/Q_{2,k}^*(c_n^*)}{dc_n} = 0$. After simplifying, this condition is equivalent to $\sqrt{*_{2,k}} = \frac{2(n-1)(2nk+n+k)c_n^* - (n-1)^2 a}{3n+1}$. Substituting this expression in the objective function of problem $(RWCP_{2,k})$, it follows that any interior stationary solution must satisfy that the objective function of problem $(RWCP_{2,k})$ evaluates to

$$\frac{(3n+1)kc_n^*}{2na+n(k-1)c_n^*} \geq \frac{3nk+k}{3nk+n} \geq \frac{6n+2}{7n} \geq \frac{2+\sqrt{2+2/n}}{4}.$$

Where the first inequality follows from the right hand side being increasing in c_n^*, and taking its lower bound $c_n^* = \frac{a}{k+1}$. The right hand side of the first equality is increasing in c_n^* if $2na > 0$, which holds for any $n \geq 1$. The second inequality follows from the left hand side being increasing in k, and taking $k = 2$. The last inequality holds for any $n \geq 1$.

Proof of Proposition 9

Proof. For any number $n \geq 2$ of Cournot competitors, facing any inverse demand function $P(Q) = a - bQ$, where $a > 0$, $b > 0$. Define the marginal cost parameter $c = \left(\frac{n+\sqrt{\frac{n(n+1)}{2}}}{3n+1}\right) a$. Consider the instance of problem (CAP) defined by the marginal cost functions $h_1(q_1) = bq_1$, $h_i(q_i) = c + bq_i$ for each $i \in \{2,\ldots,n\}$, and the budget $B = \left(\frac{(n-1)\sqrt{\frac{n(n+1)}{2}}}{(3n+1)^2}\right)\frac{a^2}{b}$. The remainder of the proof is omitted. It simply consists in checking that this instance is feasible for problem (WCP_2). Namely, that it induces that the first firm to get a positive co-payment in the optimal co-payment policy for problem (CAP) is $l = 2$. As well as evaluating its objective function, and checking it attains $\frac{2+\sqrt{2+2/n}}{4}$.