A Modified Bass Model for Product Growth in Social Networks

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Diffusion of innovations and products through social interactions has long been observed in various social systems Rogers [2003]. It often happens that a few pioneers adopt an innovation and they influence people in contact with them. Those in turn, adopt and influence their contacts, and the innovation spreads through the network as a result of these social interactions. The rapid growth and popularity of online social interactions have significantly intensified the impact of social interactions on the spread of innovation. Smartphone applications are good example of products that become widespread mainly through social interactions (word-of-mouth communication) of the users.

Understanding the temporal evolution of product/innovation adoption in a population is imperative for various managerial decisions such as pricing, revenue and inventory management. Bass first studied the timing of product adoption using a simple differential equation later known as the *Bass diffusion model* Bass [1969]. In this model at any time, the growth rate linearly depends on the fraction of population who have adopted so far. This relies on the assumption that a given agent in the population is in contact with any other agent, and thus, a non-adopter can be influenced by *all* adopters. Such global interaction, however, do not exist in many modern social networks with billions of users. In online social networks, usually each person is in contact with a small group of friends and he/she is only influenced by these people. How does product adoption evolve in such networks with limited interactions? In this paper, we develop a machinery for characterizing product growth in a large population where the structure of the underlying network of interactions belongs to in a large class of random graphs.

We use the same notion of innovation diffusion as the Bass model Bass [1969], but include the limitation on social interactions. In particular, we assume that at any time the likelihood that a non-adopter adopts a product linearly depends on the number of *its neighbors* who have previously adopted. As an equivalent model, we use a simple contact process that works as follows: agents are nodes of a graph, and an edge between two nodes means that these agent are neighbors. Initially one random node adopts the product. Later, each node contacts a randomly selected neighbor at an independent Poisson process (with a given rate). If the contacting node is an adopter and the contacted neighbor is not, then the latter adopts the product with a given probability.

Similar to Bass [1969], we are mainly concerned with timing in the regime that a fraction of the population have already adopted. We denote this phase by *major adoption regime* and we analyze the time it takes to increase the *fraction* of adopters by a constant (independent of the population size). This is in the same spirit of fluid limits in queueing theory Whitt [2002]. In particular, we show that the sample paths of the scaled adoption process (i.e., number of adopters divided by the population size) almost surely converges to a determinist function which can be viewed as the solution of a certain differential equation.

First, as a sanity check, we find the limit for the complete network (where each agent is indeed with contact with any other agent), and confirm that the fluid limit of our diffusion process coincides with the Bass model. Next and far more importantly, we establish the limit for random k-regular graphs (where k is a constant independent of the population size), and show how it differs from the result of complete graphs in different aspects. We show that the innovation spreads more slowly in random graphs compared to complete graphs. Further, unlike the complete graph, in random regular graphs, the process spreads faster in the second half of the adoption process (i.e., after reaching half of the population) compared to the first half. Finally, the differential equation associated the k-regular random graph differs from that of the complete graph in the following crucial way: the growth rate sub-linearly depends on the fraction of adopters (as opposed to linearly in the Bass model).

We note that for similar diffusion processes, differential equation approximation and (in particular, the Kurtz's theorem Kurtz [1970]) have been previously used to re-derive the Bass model when the underlying graph is assumed to be complete (e.g., Massoulie and Draief [2010]). However, such an approach can not be directly applied for other network structures. In fact, analyzing the scaled sample path of the diffusion process for general graphs is prohibitively difficult. Recent papers used concepts from mean field theory to approximate the growth rate Jackson and Rogers [2007], Young [2009] for certain class of random graphs. The basic idea of these models is to approximate the fraction of adopter neighbors of each agent by the fraction of adopters in the whole population. Our result shows such approximation also over-estimates the growth rate in random graphs, even though it does improve upon the Bass model. In order to exactly characterize the growth rate, we develop new techniques to incorporate the effect of network structure in the evolution of the diffusion process for random regular graphs. We also describe how to generalize our analysis to compute the fluid limits for random graphs with more general degree distributions (under certain condition on the distribution).

In addition to the fluid limits in the major adoption regime, we also find the limit of timing in the *early adoption regime* which refers to the phase that the first constant fraction of agents adopt the product. In this regime, we also show that compared to complete graphs, diffusion process grows more slowly in random k-regular graphs by a factor that is strictly less than one.

Our work brings together the literature on analyzing processes on random graphs, and on stochastic differential equations and fluid limits. To analyze the diffusion process on random graphs, we couple the (continues) contact process with the (discrete) graph generation process based on configuration model Wormald [1999]. Abstracting away from time, our epidemic process spreads on random graphs in the same way as the *exploration process*, defined in Molloy and Reed [1995], does. The latter was introduced to find the size of the largest connected component in random graph with given degree distributions. We use similar ideas as in Wormald [1995] to approximate the evolution of the diffusion process. When coupling with time and the contact process, we build upon these results for random graphs Wormald [1995], Molloy and Reed [1995] to compute the limit of timing in the adoption process.

There is a growing literature on pricing in social networks in the presence of social learning and word-of-mouth effects Campbell [2013], Ajorlou et al. [2014], Ifrach et al. [2012]. We believe our techniques for exactly characterizing the product growth can prove useful in designing optimal dynamic pricing mechanisms in social networks.

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