Technical Note No. 31*
Options, Futures, and Other Derivatives
John Hull

Properties of Ho–Lee and Hull–White Interest Rate Models

This note presents some of the math underlying the Ho–Lee and Hull–White one-factor models of the term structure. It follows the approach in Hull and White (1993).[1]

In a one-factor term structure, model the process for a zero-coupon bond price in the traditional risk-neutral world must have a return equal to the short rate r. Suppose that $v(t,T)$ is the volatility. Then:

$$dP(t,T) = rP(t,T)dt + v(t,T)P(t,T)dz \tag{1}$$

In this note, we will assume that v is a function only of t and T. Because the bond's price volatility declines to zero at maturity $v(t,t) = 0$.

From Ito's lemma, for any times T_1 and T_2 with $T_2 > T_1$

$$d\ln P(t,T_1) = \left[r - \frac{v(t,T_1)^2}{2}\right] dt + v(t,T_1)dz(t) \tag{2}$$

$$d\ln P(t,T_2) = \left[r - \frac{v(t,T_2)^2}{2}\right] dt + v(t,T_2)dz(t) \tag{3}$$

Define $f(t,T_1,T_2)$ as the forward rate for the period between time T_1 and T_2 as seen at time t

$$f(t,T_1,T_2) = -\frac{\ln P(t,T_2) - \ln P(t,T_1)}{T_2 - T_1}$$

From equations (2) and (3)

$$df(t,T_1,T_2) = \left[\frac{v(t,T_2)^2 - v(t,T_1)^2}{2(T_2 - T_1)}\right] dt - \left[\frac{v(t,T_2) - v(t,T_1)}{T_2 - T_1}\right] dz(t)$$

Define $R(t,T)$ as the zero rate for the period between t and T.

$$R(t,T) = f(0,t,T) + \int_0^T df(\tau,t,T)$$

so that

$$R(t,T) = f(0,t,T) + \int_0^t \left[\frac{v(\tau,T)^2 - v(\tau,t)^2}{2(T-t)}\right] d\tau - \int_0^t \left[\frac{v(\tau,T) - v(\tau,t)}{T-t}\right] dz(\tau) \tag{4}$$

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[1] See J. Hull and A. White , "Bond Option Pricing Based on a Model for the Evolution of Bond Prices," *Advances in Futures and Options Research*, 6 (1993), 1–13.

As T approaches t, $R(t,T)$ becomes $r(t)$ and $f(0,t,T)$ becomes the instantaneous forward rate, $F(0,t)$ so that

$$r(t) = F(0,t) + \int_0^t \frac{\partial}{\partial t}\frac{v(\tau,t)^2}{2}\,d\tau - \int_0^t \frac{\partial}{\partial t} v(\tau,t)\,dz(\tau)$$

or

$$r(t) = F(0,t) + \int_0^t v(\tau,t)v_t(\tau,t)d\tau - \int_0^t v_t(\tau,t)dz(\tau) \tag{5}$$

where subscripts denote partial derivatives. To calculate the process for r we differentiate with respect to t. Because $v(t,t) = 0$, this gives

$$dr = \left\{F_t(0,t) + \int_0^t [v(\tau,t)v_{tt}(\tau,t) + v_t(\tau,t)^2]d\tau - \int_0^t v_{tt}(\tau,t)dz(\tau)\right\}\,dt - v_t(\tau,t)|_{\tau=t}dz(t) \tag{6}$$

Case 1: Ho–Lee; $v(t,T) = \sigma(T-t)$

In the case, equation (5) gives

$$r(t) = F(0,t) + \sigma^2 t^2/2 - \int_0^t \sigma\,dz(\tau) \tag{7}$$

and equation (6) gives

$$dr(t) = [F_t(0,t) + \sigma^2 t]dt + \sigma\,dz$$

This is the Ho-Lee model

$$dr = \theta(t)\,dt + \sigma\,dz$$

We have proved the equation for $\theta(t)$

$$\theta(t) = F_t(0,t) + \sigma^2 t$$

Also from equation (4)

$$R(t,T) = f(0,t,T) + \sigma^2 tT/2 - \int_0^t \sigma\,dz(\tau) \tag{8}$$

From equations (7) and (8)

$$R(t,T) = f(0,t,T)+\sigma^2 tT/2+r(t)-F(0,t)-\sigma^2 t^2/2 = f(0,t,T)-F(0,t)+\sigma^2 t(T-t)/2+r(t)$$

Because

$$\ln P(t,T) = -R(t,T)(T-t)$$

It follows that

$$\ln P(t,T) = -f(0,t,T)(T-t) + F(0,t)(T-t) - \sigma^2 t(T-t)^2/2 - r(t)(T-t)$$

The forward bond price $P(0,T)/P(0,t)$ equals $e^{-f(0,t,T)(T-t)}$ so that this becomes

$$\ln P(t,T) = \ln \frac{P(0,T)}{P(0,t)} + F(0,t)(T-t) - \sigma^2 t(T-t)^2/2 - r(t)(T-t)$$

This proves:

$$P(t,T) = A(t,T)e^{-r(t)(T-t)}$$

where

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} + F(0,t)(T-t) - \frac{1}{2}\sigma^2 t(T-t)^2$$

Case 2: Hull–White; $v(t,T) = \sigma(1 - e^{-a(T-t)})/a$

In this case, equation (5) gives

$$r(t) = F(0,t) + \frac{\sigma^2}{a^2}(1 - e^{-at}) - \frac{\sigma^2}{2a^2}(1 - e^{-2at}) - \int_0^t \sigma e^{-a(t-\tau)} dz(\tau) \tag{9}$$

Equation (6) gives

$$dr(t) = \left\{ F_t(0,t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \int_0^t \sigma a e^{-a(t-\tau)} dz(\tau) \right\} dt - \sigma\, dz(t) \tag{10}$$

Substituting for

$$\int_0^t \sigma e^{-a(t-\tau)} dz(\tau)$$

from equation (9) into equation (10) we obtain

$$dr(t) = \left\{ F_t(0,t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) - ar(t) + aF(0,t) + \frac{\sigma^2}{a}(1 - e^{-at}) - \frac{\sigma^2}{2a}(1 - e^{-2at}) \right\} dt - \sigma\, dz(t)$$

or

$$dr(t) = \left\{ F_t(0,t) + aF(0,t) - ar(t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \right\} dt - \sigma\, dz(t)$$

This is the Hull–White model

$$dr(t) = (\theta(t) - ar)\, dt + \sigma\, dz$$

with

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

From equation (4)

$$R(t,T) = f(0,t,T) + \frac{\sigma^2[e^{-2a(T-t)} - e^{-2aT} - 1 + e^{-2at} - 4e^{-a(T-t)} + 4e^{-aT} + 4 - 4e^{-at}]}{4a^3(T-t)}$$

$$+\frac{\sigma(e^{-aT}-e^{-at})}{a(T-t)}\int_0^t e^{a\tau}\,dz(\tau) \tag{11}$$

From equation (9)

$$\sigma\int_0^t e^{a\tau}\,dz(\tau) = -r(t)e^{at} + F(0,t)e^{at} + \frac{\sigma^2}{a^2}(e^{at}-1) - \frac{\sigma^2}{2a^2}(e^{at}-e^{-at})$$

so that

$$R(t,T) = f(0,t,T) + \frac{\sigma^2[e^{-2a(T-t)} - e^{-2aT} - 1 + e^{-2at} - 4e^{-a(T-t)} + 4e^{-aT} + 4 - 4e^{-at}]}{4a^3(T-t)}$$

$$+\frac{(e^{-aT}-e^{-at})}{a(T-t)}\left[-r(t)e^{at} + F(0,t)e^{at} + \frac{\sigma^2}{a^2}(e^{at}-1) - \frac{\sigma^2}{2a^2}(e^{at}-e^{-at})\right]$$

Now

$$\ln P(t,T) = -R(t,T)(T-t)$$

and the forward bond price $P(0,T)/P(0,t)$ equals $e^{-f(0,t,T)(T-t)}$. After some tedious algebra we get

$$\ln P(t,T) = \ln\frac{P(0,T)}{P(0,t} + F(0,t)B(t,T) - \frac{1}{4a^3}\sigma^2(e^{-aT}-e^{-at})^2(e^{2at}-1) - r(t)B(t,T)$$

where

$$B(t,T) = \frac{1-e^{-a(T-t)})}{a}$$

showing that

$$P(t,T) = A(t,T)e^{-B(t,T)r}$$

where

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t} + F(0,t)B(t,T) - \frac{1}{4a^3}\sigma^2(e^{-aT}-e^{-at})^2(e^{2at}-1)$$

Bond Options

Consider a European option with strike price K and maturity T on a zero-coupon bond where the maturity of the bond is s. The forward price of the bond underlying the option as seen at time t, $F_B(t, T, s)$, is

$$F_B(t, T, s) = \frac{P(t, s)}{P(t, T)}$$

Using the results in equations (2) and (3) we get

$$d \ln F_B(t, T, s) = \frac{v(t, T)^2 - v(t, s)^2}{2}\, dt + [v(t, s) - v(t, T)]\, dz$$

This shows that the $P(T, s) = f_B(T, T, s)$ is lognormal when $v(t, T)$ is function only of t and T. The variance $\ln P(T, s)$ is then

$$\sigma_P^2 = \int_0^T [v(t, s) - v(t, T)]^2 dt$$

In the case of Ho-Lee $v(t, T) = \sigma(T - t)$ and $\sigma_P^2 = \sigma^2(s - T)^2 T$. In Hull-White $v(t, T) = \sigma B(t, T)$ so that

$$\sigma_P^2 = \sigma^2 \int_0^T [B(t, s) - B(t, T)]^2\, dt = \frac{\sigma^2}{2a^3}[1 - e^{-a(s-T)}]^2(1 - e^{-2aT})$$

In both cases bond options can be valued using Black's model. The average variance rate of the forward bond price is σ_P^2/T. This leads to the results for bond options in the text.