USING HULL-WHITE INTEREST RATE TREES

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The Hull-White interest rate tree-building procedure was first outlined in the Fall 1994 issue of the Journal of Derivatives. *It is becoming widely used by practitioners. This procedure is appropriate for models where there is some function $x = f(r)$ of the short rate r that follows a mean-reverting arithmetic process. It can be used to implement the Ho-Lee model, the Hull-White model, and the Black-Karasinski model. Also, it is a tool that can be used for developing a wide range of new models.*

In this article we provide more details on the ways Hull-White trees can be used. We discuss the analytic results available when $x = r$, and make the point that it is important to distinguish between the Δt-period rate over one time step on the tree and the instantaneous *short rate that is used in some of these analytic results. We provide an example of the implementation of the model using market data. We show how the tree can be designed so that it provides an exact fit to the initial volatility environment (but at the same time explain why we do not recommend this approach). We also discuss how to deal with such issues as variable time steps, cash flows that occur between nodes, barrier options, and path-dependence.*

In Hull and White [1994a], we describe a procedure for constructing trinomial trees for one-factor yield curve models of the form:

$$dx = [\theta(t) - ax]\,dt + \sigma dz \tag{1}$$

where $x = f(r)$ is some function of the short rate r, $\theta(t)$ is a function of time chosen so that the model provides an exact fit to the initial term structure of interest rates, and a and σ are constants.

The model can be written

$$dx = a\left[\frac{\theta(t)}{a} - x\right]dt + \sigma dz$$

This shows that, at any given time, x reverts toward the value $\theta(t)/a$ at rate a. Its variance rate per unit time is σ^2.

When $f(r) = r$, the model reduces to the Hull-White [1990] model:

$$dr = a[\theta(t) - r]\,dt + \sigma dz \tag{1A}$$

The attraction of the Hull-White model is its analytic tractability. As shown in Hull and White [1990, 1994a], bonds and European options at some future time t can be valued analytically in terms of the initial term structure and the value of r at time t. When $f(r) = \log(r)$, and a and σ are allowed to be functions of time, the model becomes that of Black and Karasinski [1991]. When $f(r) = \log(r)$, $a(t) = -\sigma'(t)/\sigma(t)$, and $\sigma'(t) = \partial\sigma/\partial t$, the model becomes the Black, Derman, and Toy [1990] model.

Construction of the Hull-White tree occurs in two stages. The first stage involves defining a new variable x^* obtained from x by setting both $\theta(t)$ and the initial value of x equal to zero. The process for x^* is:

$$dx^* = -ax^*dt + \sigma dz \tag{2}$$

We construct a tree for x^* that has the form shown in Exhibit 1. The central node at each time step has $x^* = 0$. The vertical distance between the nodes on the tree is set equal to $\Delta x^* = \sqrt{3V}$, where V is the variance of the change in x in time Δt, the length of each time step. The probabilities at each node are chosen to match the mean and standard deviation of the change in x^* for the process in Equation (2).[1]

Defining the expected change in x^* as Mx^*, at node $j\Delta x^*$ the up-, middle-, and down-branching probabilities are

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2}$$

$$p_m = \frac{2}{3} - j^2M^2$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2} \tag{3A}$$

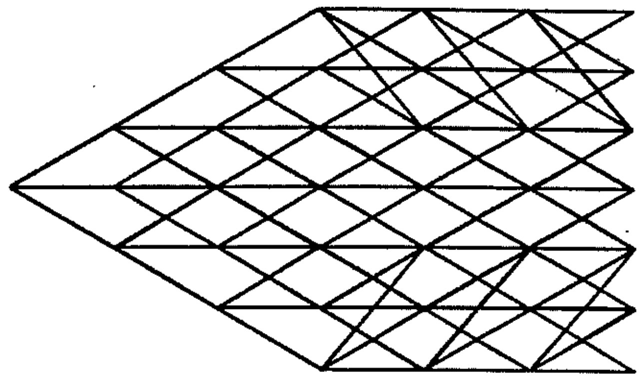
As indicated in Exhibit 1, we cope with mean reversion by allowing the branching to be non-standard at the edge of the tree. At the top edge of the tree, where the branching is non-standard, the modified probabilities become

$$p_u = \frac{7}{6} + \frac{j^2M^2 + 3jM}{2}$$

$$p_m = -\frac{1}{3} - j^2M^2 - 2jM$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 + jM}{2} \tag{3B}$$

EXHIBIT 1
THE INITIAL TREE FOR X^* (SETTING $\theta(t) = 0$ AND $X(0) = 0$)



and at the bottom edge of the tree where the branching is non-standard, the modified probabilities become

$$p_u = \frac{1}{6} + \frac{j^2M^2 - jM}{2}$$

$$p_m = -\frac{1}{3} - j^2M^2 + 2jM$$

$$p_d = \frac{7}{6} + \frac{j^2M^2 - 3jM}{2} \tag{3C}$$

The second stage in construction of the tree involves forward induction. We work forward from time zero to the end of the tree, adjusting the location of the nodes at each time step so as to match the initial term structure. This produces a tree of the form shown in Exhibit 2. The size of the displacement is the same for all nodes at a particular time t, but is not usually the same for nodes at two different times. The effect of this second stage is to convert a tree for x^* into a tree for x.

The full details of the tree-building procedure are given in Hull and White [1994a]. In Hull and White [1994b], we describe modeling two interest rates simultaneously and using the tree-building technology to construct two-factor models of a single term structure.

The purpose of this article is to provide more details on the basic Hull-White tree-building procedure. We discuss how to use analytic results when f(r)

= r. We provide sample results based on a real yield curve that readers can use to test their own implementation of the model. We show how the tree-building procedure can be used for models such as Black and Karasinski [1991] where a and σ are functions of time, but point out some pitfalls of these models. We also discuss how the length of the time step can be changed, how cash flows that occur between time steps can be handled, and so on.

I. ANALYTIC RESULTS

Bond Prices

When f (r) = r, the model in Equation (1) is analytically very tractable. For example, as shown in Hull and White [1990, 1994a]:

$$P(t, T) = A(t, T)e^{-B(t,T)r} \tag{4}$$

where P(t, T) is the price at some time t of a zero-coupon bond maturing at time T, A and B are functions only of t and T, and r is the short-term rate of interest at time t. The function A is determined from the initial values of the discount bonds, P(0, T):

$$A(t, T) = \frac{P(0, T)}{P(0, t)}\exp[B(t, T)\, F(0, t) - \sigma^2 B(t, T)^2(1 - e^{-2at})/(4a)]$$

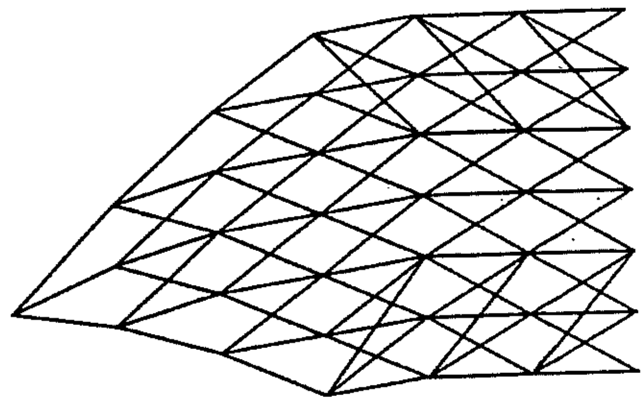
$$B(t, T) = (1 - e^{-a(T-t)})/a \tag{5}$$

F (0, t) is the instantaneous forward rate that applies to time t as observed at time zero. It can be computed from the initial price of a discount bond as F (0, t) = $-\partial \log[P(0, t)]/\partial t$.

The variable r in Equation (4) is the instantaneous short rate, while the interest rates on the Hull-White tree are Δt-period rates. The two should not be assumed to be interchangeable. Let R be the Δt period rate at time t, and r be the instantaneous rate at time t. Using Equation (4):

$$e^{-R\Delta t} = A(t, t + \Delta t)\, e^{-B(t,t+\Delta t)r}$$

EXHIBIT 2
THE FINAL TREE FOR X



so that

$$r = \frac{R\Delta t + \log A(t, t + \Delta t)}{B(t, t + \Delta t)} \tag{6}$$

To calculate points on the term structure, given the Δt period rate R at a node of the Hull-White tree, it is first necessary to use Equation (6) to get the instantaneous short rate, r. Equation (4) can then be used to determine rates for longer maturities. When this procedure is followed, it can be shown that the prices of discount bonds that are computed are independent of the forward rate, F (0, t).[2]

Expected Future Rates

Inspection of Equations (1) and (2) shows that x(t) and $x^*(t)$ differ only by some function of time. Define this difference as

$$\alpha(t) = x(t) - x^*(t) \tag{7}$$

This is the difference between the location of comparable nodes in the x and x^* trees at time t. In particular, it is the difference between the central or expected values of x and x^* at time t, and since the expected value of x^* is zero, α(t) can be interpreted as the expected value of x(t). As Kijima and Nagayama [1994] and Pelsser [1994] point out, α(t) can be calculated analytically for the model where f (r) = r.

Differentiating Equation (7), it follows from Equations (1) and (2) that

$$\frac{\partial\alpha(t)}{\partial t} = \theta(t) - a\alpha(t)$$

or

$$\alpha(t) = \exp\left\{-at\left[r(0) + \int_0^t \theta(q)\, e^{aq} dq\right]\right\}$$

Substituting the analytic expression for $\theta(t)$ given in Hull and White [1990, 1994a], this reduces to

$$\alpha(t) = F(0, T) + \frac{\sigma^2}{2a^2(1 - e^{-at})^2} \tag{8}$$

Use of the analytic expression for α to determine the location of the central nodes in the tree avoids the need to obtain them from forward induction.[3] The resulting tree, however, does not provide an exact fit to the initial term structure. This is because the tree is a discrete representation of the underlying continuous stochastic process. The advantage of the forward induction procedure is that the initial term structure is always matched exactly by the tree itself.

EXHIBIT 3
THE DM ZERO-COUPON YIELD CURVE, JULY 8, 1994

Maturity	Days	Rate
3 days	3	5.01772
1 month	31	4.98284
2 months	62	4.97234
3 months	94	4.96157
6 months	185	4.99058
1 year	367	5.09389
2 years	731	5.79733
3 years	1,096	6.30595
4 years	1,461	6.73464
5 years	1,826	6.94816
6 years	2,194	7.08807
7 years	2,558	7.27527
8 years	2,922	7.30852
9 years	3,287	7.39790
10 years	3,653	7.49015

II. AN EXAMPLE

To give an example of the implementation of the model, we use the data in Exhibit 3. These data, which are for the DM yield curve on July 8, 1994, were kindly provided to us by Antoon Pelsser of ABN Amro Bank. Data points for maturities between those indicated are generated using linear interpolation.

The zero curve is used to price a three-year (= 3 × 365-day) put option on a zero-coupon bond that will pay $100 in nine years (= 9 × 365 days).[4] Interest rates are assumed to follow the Hull-White [Equation (1A)] model. The strike price is $63, and the parameters a and σ are chosen to be a = 0.1, and σ = 0.01. These two parameters determine the volatility of the discount bond for option pricing purposes. The values chosen are roughly representative of the values observed in the market. The tree is constructed out to the end of the life of the option. The zero-coupon bond prices at the final nodes are calculated analytically as described above.

Suppose you want to construct a three-step tree. First, we must determine the time and rate step sizes, and where non-standard branching (if any) takes place. The size of the time step is Δt = 365 days = 1.0 years.

As shown in Hull and White [1994a], the expected change in r^* and the variance of the change in r^* in time Δt are given by

$$E[dr^*] = Mr^* = (e^{-a\Delta t} - 1)\, r^*$$

$$Var[dr^*] = V = \sigma^2(1 - e^{-2a\Delta t})/2a$$

For the given parameter values, M = −0.095162582 and $\sqrt{V}$ = 0.009520222. Since the step size $\Delta r = \sqrt{3V}$, Δr = 0.016489508. Finally, as shown in Hull and White [1994a], non-standard branching takes place at nodes $\pm j^*$ where j^* is the smallest integer greater than −0.184/M. In this case, j^* is 2. The data defining the initial tree are shown in Exhibit 4.

EXHIBIT 4
DATA DEFINING A THREE-STEP TREE IN r*

j	Rate = $j\Delta r$	p_u	p_m	p_d	Equation
2	0.032979	0.899291	0.011093	0.089616	3B
1	0.016490	0.123613	0.657611	0.218776	3A
0	0.0	0.166667	0.666667	0.166667	3A
−1	−0.016490	0.218776	0.657611	0.123613	3A
−2	−0.032979	0.089616	0.011093	0.899291	3C

The rates at each node in the tree at each time step are now shifted up by some amount, α, chosen so that the revised tree correctly prices discount bonds. Since there are nodes at the one-, two-, and three-year points, we need the discount bond prices corresponding to these dates as well as the four-year price, one time step beyond the option maturity. When the option price is calculated, the nine-year bond price will be required as well.

This information, interpolated from the data in Exhibit 3, is shown in Exhibit 5. Exhibit 5 also shows the value of α required to fit the bond prices at each time step. An efficient procedure for implying the value of α is given in Hull and White [1994a]. For reference purposes, the instantaneous forward rate and the instantaneous values of α [based on Equation (8)] are also shown.

Combining the αs from Exhibit 5 with the rates and probabilities in Exhibit 4 produces the complete tree. The tree is shown in Exhibit 6, which shows the Δt period rates at each node of the tree and the probabilities of branching from one node to the next.

Exhibit 7 shows how this tree can be used to compute the price of a two-year discount bond. At each step, the bond price is computed as the discounted value of the expected value at the next time-step. Calculations of the type shown in Exhibit 7 are used to determine what value of α is needed at each time

EXHIBIT 5
AMOUNT, α, BY WHICH INTEREST RATES AT EACH TIME STEP MUST BE RAISED TO REPLICATE BOND PRICES COMPUTED FROM ZERO-COUPON DISCOUNT RATES

Time Step i	t = $i\Delta t$ Years	Zero Rate (%)	Discount Bond Price	α (%)	Forward Rate (%)	$\alpha(t)$ – Equation (8) (%)
0	0.0	5.017720	1.000000	5.09275	5.017720	5.017720
1	1.0	5.092755	0.950348	6.50257	5.299942	5.304470
2	2.0	5.795397	0.890557	7.33932	7.206143	7.222572
3	3.0	6.304557	0.827673	8.05381	7.830417	7.864004
4	4.0	6.733466	0.763885			
	9.0	7.397410	0.513879			

EXHIBIT 6
THE FOUR TIME STEPS IN THE INTEREST RATE TREE

	Transition Probabilities			Node Rates, R, (%)			
j	p_u	p_m	p_d	i = 0	i = 1	i = 2	i = 3
2	0.8993	0.0111	0.0896			10.6372	11.3517
1	0.1236	0.6576	0.2188		8.1515	8.9883	9.7028
0	0.1667	0.6667	0.1667	5.0928	6.5026	7.3393	8.0538
−1	0.2188	0.6576	0.1236		4.8536	5.6904	6.4049
−2	0.0896	0.0111	0.8993			4.0414	4.7559

The probabilities of transiting from node (i, j) to nodes (i + 1, j + 1), (i + 1, j), and (i + 1, j − 1) are normally $p_u(j)$, $p_m(j)$, and $p_d(j)$, respectively. When j = ±2, the alternative branching schemes are used.

EXHIBIT 7

COMPUTING THE PRICE OF A BOND THAT PAYS $1 AT TIME 2Δt (TWO YEARS)

	Transition Probabilities			Bond Price		
j	p_u	p_m	p_d	i = 0	i = 1	i = 2
2	0.8993	0.0111	0.0896			1.0
1	0.1236	0.6576	0.2188		0.9217	1.0
0	0.1667	0.6667	0.1667	0.8906	0.9370	1.0
–1	0.2188	0.6576	0.1236		0.9526	1.0
–2	0.0896	0.0111	0.8993			1.0

Each value is calculated as $v_{i,j} = (p_u v_{i+1,j+1} + p_m v_{i+1,j} + p_d v_{i+1,j-1}) \exp(-R_{i,j} \Delta t)$.

step in order to replicate the discount bond prices.

Exhibit 8 shows the calculations required to compute the discount bond prices at the option maturity, three years. Finally, Exhibit 9 shows the discounting of the option value back through the tree.

The results of pricing this put option for trees of different size are shown in Exhibit 10. This example provides a good test of implementation of the model because the gradient of the zero curve changes sharply immediately after the expiration of the option. Small errors in the construction and use of the tree are liable to have a big effect on the option values obtained. For example, when 100 time steps are used, the value of the option is reduced by about $0.25 if the Δt-period rate is assumed to be the instantaneous rate.

EXHIBIT 8

COMPUTING THE OPTION PAYOFF AT EACH TERMINAL NODE (i = 3) ON THE TREE

j	Δt-Period Rate, R	Instantaneous Rate, r	Bond Price	Option Payoff
2	0.113517	0.113206	0.529196	10.080445
1	0.097028	0.095878	0.572229	5.777133
0	0.080538	0.078550	0.618761	1.123884
–1	0.064049	0.061222	0.669078	0.0
–2	0.047559	0.043895	0.723486	0.0

The Δt-period rate, R, is the rate that applies from three to four years. The instantaneous rate, r, is computed using Equation (6). The forward rate at time 3 is computed to be 0.078304. On the basis of this, Equation (5) gives

$$A(3, 4) = 0.994229, A(3, 9) = 0.881944,$$
$$B(3, 4) = 0.951626, B(3, 9) = 4.511884$$

The bond price, P(3, 9), is computed with Equation (4), and the option payoff is 100 Max[0.63 –P(3, 9), 0].

III. MAKING VOLATILITY PARAMETERS TIME-DEPENDENT

When a and σ are functions of time, the model in Equation (1) becomes

$$dx = [\theta(t) - a(t)\, x]\, dt + \sigma(t)\, dz \qquad (9)$$

The three functions of time in this diffusion equation each play a separate role. The function θ(t) is chosen so that the prices of all discount bonds are matched at the initial time. The other two functions provide two extra degrees of freedom that allow us to match the initial volatility of all zero-coupon rates and the volatility of the short rate at all future times.

The tree can then be tuned to price not only the zero-coupon bonds, but also a set of interest rate derivatives at their current market prices. The initial volatility of all rates depends on σ(0) and a(t). The volatility of the short rate at future times is determined by σ(t). Unless σ(t) and a(t) are constants, the volatility term structure is non-stationary.

Our tree-building procedure can be extended to accommodate the model in Equation (9). Analogously to the constant a and σ case, we first build a tree for x^* where

$$dx^* = -a(t)x^* dt + \sigma(t)dz$$

We first choose the times at which nodes will be placed, $t_0, t_1, t_2, \ldots, t_n$, where $t_0 = 0$, and $t_i = i\Delta t$

EXHIBIT 9
DISCOUNTING THE OPTION PRICE BACK THROUGH THE TREE

	Transition Probabilities			Time Step, i			
j	p_u	p_m	p_d	0	1	2	3
2	0.8993	0.0111	0.0896			8.2987	10.0804
1	0.1236	0.6576	0.2188		4.1977	4.8362	5.7771
0	0.1667	0.6667	0.1667	1.8734	1.7854	1.5910	1.1239
−1	0.2188	0.6576	0.1236		0.4885	0.2323	0.0000
−2	0.0896	0.0111	0.8993			0.0967	0.0000
			α (%)	5.0928	6.5026	7.3393	8.0538

At the third step the option value is as given in Exhibit 8. The computed value at earlier steps is

$$v_{i,j} = (p_u v_{i+1,j+1} + p_m v_{i+1,j} + p_d v_{i+1,j-1}) \exp(-R_{i,j}\,\Delta t)$$

where $R_{i,j}$, the rate at node j and time step i, is $\alpha_i + j\Delta r$. Note that when $j = \pm 2$, non-standard branching applies. When $j = 2$ the computed value is $v_{i,j} = (p_u v_{i+1,j} + p_m v_{i+1,j-1} + p_d v_{i+1,j-2}) \exp(-R_{i,j}\,\Delta t)$, and when $j = -2$ the computed value is $v_{i,j} = (p_u v_{i+1,j+2} + p_m v_{i+1,j+1} + p_d v_{i+1,j}) \exp(-R_{i,j}\,\Delta t)$.

for $i = 0, \ldots, n$. The vertical (x^* dimension) spacing between adjacent nodes at time t_{i+1} is then set equal to $\sqrt{3V_i}$ where

$$V_i = \sigma(t_i)^2(1 - e^{-2a(t_i)\Delta t})/2a(t_i)$$

Suppose that the value of x^* at the jth node at time t_i is $x^*_{i,j}$. The mean and standard deviation of x^* at time t_{i+1} conditional on $x^* = x^*_{i,j}$ at time t_i are approximately $x^*_{i,j} + M_i x^*_{i,j}$ and $\sqrt{V_i}$, where

$$M_i = (e^{-a(t_i)\Delta t} - 1)$$

We match these by branching from $x^*_{i,j}$ to one of $x^*_{i+1,k-1}$, $x^*_{i+1,k}$, and $x^*_{i+1,k+1}$, where k is chosen so that $x^*_{i+1,k}$ is as close as possible to $x^*_{i,j} + M_i x^*_{i,j}\Delta t$. We then calculate the displacements, $\alpha(t)$, necessary for the tree to match the initial term structure.

The a(t) and σ(t) can be set in advance of the numerical procedure. Alternatively, it is not difficult to devise a numerical procedure that chooses a(t) and σ(t) so that the initial prices of caps or swap options (or both) are matched. When used for x = log(r), this type of tree-building procedure has the advantage over Black and Karasinski [1991] that the length of the time step is under the control of the user.

It seems appealing to take advantage of all the degrees of freedom in a model to fit initial market data exactly. The resulting non-stationarity in the volatility term structure, however, may have many untoward and unexpected effects. To illustrate this, we use the x = r model:

$$dr = [\theta(t) - a(t)\,r]\,dt + \sigma(t)\,dz$$

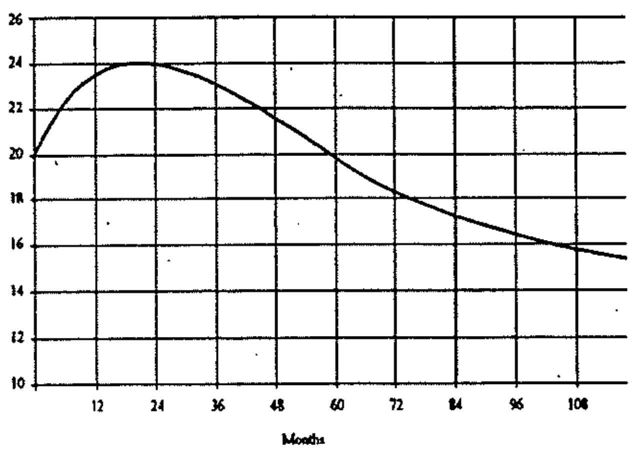
and show the effect of matching cap prices.

Caps are usually priced using Black's model,

EXHIBIT 10
VALUE OF A THREE-YEAR PUT OPTION ON A NINE-YEAR, \$100, ZERO-COUPON BOND — STRIKE PRICE \$63; VOLATILITY PARAMETERS a = 0.1 AND σ = 0.01

Steps	Tree-Based Value	Analytic Value
10	1.8491	1.8093
30	1.8179	1.8093
50	1.8060	1.8093
100	1.8128	1.8093
200	1.8089	1.8093
500	1.8090	1.8093

EXHIBIT 11
BLACK'S VOLATILITY FOR AT-THE-MONEY CAPLETS RESET MONTHLY



under which the price at time zero of a caplet expiring at T on a rate that applies from T to T + τ is

$$C = \tau P e^{-R(T+\tau)}[F(T, T+\tau)N(d_1) - XN(d_2)]$$

where P is the notional principal, R is the zero-coupon rate with a maturity T + τ, F(T, T + τ) is the forward rate for the period T to T + τ, X is the cap rate, and

$$d_1 = \frac{\log(F(T, T+\tau)/X}{v(T)\sqrt{T}} + \frac{v(T)\sqrt{T}}{2}$$

$$d_2 = d_1 - v(T)\sqrt{T}$$

where v(T) is the volatility for the caplet expiring at T.

The data set that we use for calibration consists of the market prices of at-the-money caps that are reset monthly (τ = 1 month). The particular v(T) function we assume for illustration purposes is shown in Exhibit 11.[5] This has a similar shape to the v(T) function commonly observed in the market. We assume the term structure is flat at 7% continuously compounded.

In order to match the Black volatilities we first use them in conjunction with Black's model to calculate caplet prices. We then match the caplet prices in two ways:

1. We fix the short rate standard deviation, σ, and allow the reversion rate, a, to be a function of time.
2. We fix a and allow σ to be a function of time.

Exhibit 12 shows the value of a(t) required to fit the market data when σ is fixed at 1.4% and the value of σ(t) required to fit the market data when a is fixed at 5%.[6] It can be seen that the implied a(t) and σ(t) exhibit severe non-stationarity. Although by construction this non-stationarity leads to caplets being priced correctly, it is liable to lead to unacceptable results when used to price other instruments.

Any instrument whose price depends on the future volatility structure, rather than today's volatility structure, is liable to be mispriced by a model with time-dependent volatility parameters. One example of such a security is an American-style call option where the decision to exercise at some future date depends on the volatility structure at that date. Another example is a caption, an option to buy a cap, where the decision to exercise the option at expiration depends on the value of the cap at that time.

This example illustrates the sort of problems that can arise when a model is implemented in such a way that the volatility structure is not stationary. It is a

EXHIBIT 12
VALUE OF a(t) WHEN σ = 1.4% (LEFT-HAND SCALE), AND VALUE OF σ(t) WHEN a = 5% (RIGHT-HAND SCALE) REQUIRED TO REPLICATE CAPLET PRICES COMPUTED FROM BLACK VOLATILITIES IN EXHIBIT 11

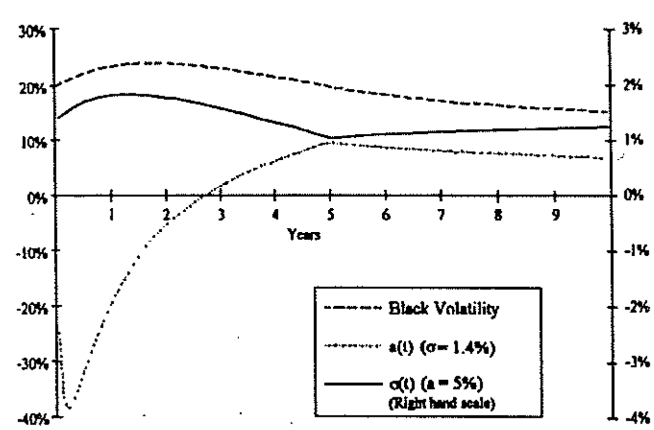
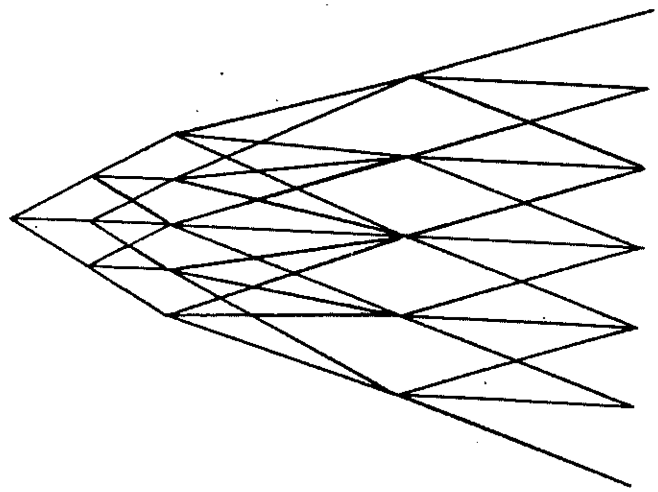


EXHIBIT 13
THE TREE FOR X^* WHEN LENGTH OF TIME STEP CHANGES



problem that afflicts all Markov interest rate models including the Black, Derman, and Toy and Black and Karasinski models.

By fitting a one-factor Markov interest rate model to today's option prices, we make it reflect the initial volatility structure exactly, but we are also unwittingly making a statement about how the volatility term structure will evolve in the future. Using all the degrees of freedom in the model to fit the volatility exactly constitutes an overparameterization of the model. It is our opinion that there should be no more than one time-varying parameter used in Markov models of the term structure evolution, and that this should be used to fit the initial term structure.

IV. OTHER ISSUES

There are a number of other practical issues to consider when implementing Hull-White trees for valuing interest rate derivatives. We review some of them and indicate how to handle them.

In our description of the tree-building procedure in Hull and White [1994a], we assume that the length of the time step is constant. In practice, it is sometimes desirable to change the length of the time step.[7] Changing the length of the time step is straightforward. When drawing the tree for x^*, we first choose the times at which nodes will be placed, t_0, t_1, t_2, ..., t_n, where $t_0 = 0$. Defining $\Delta t_i = t_{i+1} - t_i$ for $i = 0, \ldots, n-1$, the vertical (x^* dimension) spacing between adjacent nodes at time t_{i+1} is then set equal to $\sqrt{3V_i}$ where

$$V_i = \sigma^2(1 - e^{-2a\Delta t_i})/2a$$

From this point, construction is similar to the procedure followed when the volatility parameters are a function of time. Suppose that the value of x^* at the jth node at time t_i is $x^*_{i,j}$. The mean and standard deviation of x^* at time t_{i+1} conditional on $x^* = x^*_{i,j}$ at time t_i are approximately $x^*_{i,j} + M_i x^*_{i,j}$ and, where

$$M_i = (e^{-a\Delta t_i} - 1)$$

We match these by branching from $x^*_{i,j}$ to one of $x^*_{i+1,k-1}$, $x^*_{i+1,k}$, or $x^*_{i+1,k+1}$, where k is chosen so that $x^*_{i+1,k}$ is as close as possible to $x^*_{i,j} + M_i x^*_{i,j} \Delta t_i$. Note that whenever the size of the time step changes, $\Delta t_i \neq \Delta t_{i+1}$, the vertical ($x^*$ dimension) spacing between nodes increases by $\sqrt{\Delta t_{i+1}/\Delta t_i}$. This means that the branching is non-standard at points when the length of the time step changes.

Exhibit 13 illustrates the tree that is constructed when the time step increases by a factor of three after two time steps.

The tree for x is constructed from the tree for x^* to match the initial zero-coupon yield curve as described in Hull and White [1994a]. Note that, when the length of the time step changes from Δt_i to Δt_{i+1}, the interest rates considered at the nodes automatically change from the Δt_i period rates to the Δt_{i+1} rates.

Another issue in construction of the tree concerns cash flows that occur between nodal dates. Suppose a cash flow occurs at time τ when the immediately preceding nodal date is t_i and the immediately following nodal date is t_{i+1}. One approach is to discount the cash flow from time τ to the nodes at time t_i using estimates of the $\tau - t_i$ rates prevailing at the nodes at time t_i.[8] Another approach is to assume that a proportion $(\tau - t_i)/(t_{i+1} - t_i)$ of the cash flow occurs at time t_{i+1} while the remainder occurs at time t_i.[9] A final approach is to avoid the problem altogether by changing the length of the time step so that every payment date is also a nodal date.

Barrier options present a further problem in use of the tree, because convergence tends to be slow when nodes do not lie exactly on barriers. In the case of an interest rate option, the barrier is typically expressed in terms of a bond price or a particular rate. When x = r, analytic results can be used to express the barrier as a function of the Δt-period rate. Nonstandard branching can then be used to ensure that nodes always lie on the barrier. Ritchken [1995] describes such an approach, and shows that a substantial improvement in performance is possible with it.

An alternative approach that has more general applicability is to extend the idea suggested by Derman et al. [1995] to interest rate trees. This approach involves using a procedure to correct values of the derivative calculated at nodes close to a barrier.

A final problem in the use of interest rate trees is path-dependence. This can sometimes be handled in the way described by Hull and White [1993]. The requirements for the Hull-White method to work are:

1. The value of the derivative at each node must depend on just one function of the path for the short rate r (e.g., the maximum, minimum, or average value).
2. In order to update the path function as we move forward through the tree we need to know only the previous value of the function and the new value of r.

Hull and White show how the approach can be used for index amortizing swaps and mortgage-backed securities. The relevant path function in each case is the remaining principal.

V. SUMMARY

The Hull-White tree-building procedure is a flexible approach to constructing trees for a wide range of different one-factor models of the term structure. The tree is constructed to be exactly consistent with the initial term structure.

In this article we show how to extend the basic procedure presented in our earlier work. Some of these extensions involve the use of analytic results, and some involve changing the geometry of the tree to reflect special features of the derivative under consideration.

We have devoted some time to a discussion of what happens when the volatility parameters are made time-dependent. It not difficult to extend the Hull-White tree to incorporate time-dependent parameters so that the prices of caps or swap options (or both) are matched, but this is liable to result in unacceptable assumptions about the evolution of volatilities.

ENDNOTES

[1]The expected value and variance of the change in x^* over some time Δt are

$$E[dx^*] = Mx^* = (e^{-a\Delta t} - 1)\, x^*; \text{ and}$$
$$Var[dx^*] = V = \sigma^2(1 - e^{-2a\Delta t})/2a$$

[2]Since the forward rate is computed from the first derivative of the yield curve, it is very sensitive to the exact shape of the yield curve. Slight variations in the yield curve create large changes in the computed forward rate. If the computed bond price had depended on the forward rate, the results would be very sensitive to exactly how one computes the yield curve.

[3]Forward induction is always necessary when $f(r) \neq r$, because there are no analytic results in that case.

[4]The fundamental unit of time in this example is one day. For convenience, we define one year as 365 days, which is approximately the length of a real year, and quote rates and volatilities per year. The data in Exhibit 3 are quoted on this basis. Thus the ten-year rate of 7.49015% is actually a rate of 0.0205210% per day. This rate applies for 3,653 days or about 10.0082 years. This convention may seem cumbersome, but is necessary to avoid the ambiguity associated with the variable length of a calendar year.

[5]This volatility curve is

$$v(T) = [1 + bT + c\,(1 - e^{-dT})]\, v(0)$$

for $T \leq 5$, $b = -0.1$, $c = 0.5$, $d = 0.8$, and $v(0) = 0.2$. The curve is extended beyond $T = 5$ by assuming that the gradient of $v(T)T$ when $T > 5$ equals its gradient when $T = 5$.

[6]The choices of the fixed value for σ and the fixed value for a are arbitrary, although the implied values of $a(t)$ and $\sigma(t)$ are representative of the type of non-stationarity that results from the given volatility structure. The best fixed value of σ (or a) to use might be the one that minimizes the variance of the implied $a(t)$ [or $\sigma(t)$].

[7]Suppose the lognormal model is used to value a European six-month option on a five-year bond. It might be appropriate to use a longer Δt between six months and five years than during the first six months. This is because the part of the tree between six months and five years is used only to value the underlying bond.

[8]In the case of the Hull-White x = r model, these rates can be calculated analytically.

[9]This approach has the effect of apportioning the cash flow to nodal dates, while ensuring that the expected time when the cash flow occurs is correct.

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