Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models

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In the last issue of this Journal we explained a new procedure for building trinomial trees for one-factor no-arbitrage models of the term structure. The procedure is appropriate for models where there is some function, x, of the short rate, r, that follows a mean-reverting arithmetic process. It is relatively simple and computationally more efficient than previously proposed procedures.

In this article we extend the new tree-building procedure in two ways. First, we show how it can be used to model the yield curves in two different countries simultaneously. Second, we show how to develop a variety of Markov two-factor models.

In order to model the yield curves in two different countries simultaneously, we first build a tree for each of the yield curves separately. The trees are then adjusted so that they are both constructed from the viewpoint of a risk-neutral investor in the country in which the cash flows are realized. We then combine the two trees on the assumption that there is no correlation. In the final step we induce the required amount of correlation by adjusting the probabilities on the branches emanating from each node.

The two-factor Markov models of the term structure that we consider incorporate a stochastic reversion level. They have the property that they can accommodate a much wider range of volatility structures than the one-factor models considered in our earlier article. In particular, they can give rise to a volatility "hump" similar to that observed in the cap market. We show that one version of the two-factor models is somewhat analytically tractable.

The most general approach to constructing models of the term structure is the one suggested by Heath, Jarrow, and Morton [1992]. This involves specifying the volatilities of all forward rates at all times. The expected drifts of forward rates in a risk-neutral world are calculated from their volatilities, and the initial values of the forward rates are chosen to be consistent with the initial term structure.

Unfortunately, the models that result from the

Heath, Jarrow, and Morton approach are usually non-Markov, meaning that the parameters of the process for the evolution of interest rates at a future date depend on the history of interest rates as well as on the term structure at that date. To develop Markov one-factor models, an alternative approach has become popular. This involves specifying a Markov process for the short-term interest rate, r, in terms of a function of time, $\theta(t)$. The function of time is chosen so that the model exactly fits the current term structure.

Hull and White [1994] consider models of the short rate, r, of the form

$$dx = [\theta(t) - ax]dt + \sigma dz \tag{1}$$

where $x = f(r)$ for some function f; a and σ are constants; and $\theta(t)$ is a function of time chosen so that the model provides an exact fit to the initial term structure. This general family contains many of the common one-factor term structure models as special cases. When $f(r) = r$ and $a = 0$, the model reduces to

$$dr = \theta(t)dt + \sigma dz$$

This is the continuous time limit of the Ho and Lee [1986] model. When $f(r) = r$ and $a \neq 0$, the model becomes

$$dr = [\theta(t) - ar]dt + \sigma dz$$

and is a version of the Hull and White [1990] extended-Vasicek model. When $f(r) = \log(r)$, the model is

$$d\log(r) = [\theta(t) - a\log(r)]dt + \sigma dz$$

which is a version of Black and Karasinski [1991].

In this article we extend the ideas in Hull and White [1994] to show how the yield curves in two different countries can be modeled simultaneously. We also show how the Hull-White approach can be used to develop a variety of different Markov two-factor models of the term structure.

I. THE HULL-WHITE ONE-FACTOR TREE-BUILDING PROCEDURE

The first step for building a tree for the model in Equation (1) is to set $\theta(t) = 0$ so that the model becomes

$$dx = -axdt + \sigma dz \tag{2}$$

A time step, Δt, is chosen, and a trinomial tree is constructed for the model in Equation (2) on the assumption that the initial value of x is zero. The spacing between the x-values on the tree, Δx, is set equal to $\sigma\sqrt{3\Delta t}$. The probabilities on the tree are chosen to match the mean and the standard deviation of changes in x in time Δt.

The standard branching process on a trinomial tree is "one up, no change, or one down." This means that a value $x = j\Delta x$ at time t leads to a value of x equal to one of $(j + 1)\Delta x$, $j\Delta x$, or $(j - 1)\Delta x$ at time $t + \Delta t$. Due to mean reversion, for large enough positive and negative values of j, when this branching process is used, it is not possible to match the mean and standard deviation of changes in x with positive probabilities on all three branches. To overcome this problem, a non-standard branching process is used in which the values for x at time $t + \Delta t$ are either $(j + 2)\Delta x$, $(j + 1)\Delta x$, and $j\Delta x$, or $j\Delta x$, $(j - 1)\Delta x$, and $(j - 2)\Delta x$.

Once the tree has been constructed so that it corresponds to Equation (2), it is modified by increasing the values of x for all nodes at time $i\Delta t$ by an amount α_i. The α_is are chosen inductively so that the tree is consistent with the initial term structure. This produces a tree corresponding to Equation (1). When the Ho-Lee or Hull-White model is used so that $x = r$, the values of α_i can be calculated analytically. In other cases an iterative procedure is necessary.

This procedure provides a relatively simple and numerically efficient way of constructing a trinomial tree for x and implicitly determining $\theta(t)$. The tree has the property that the central node at each time is the expected value of x at that time.

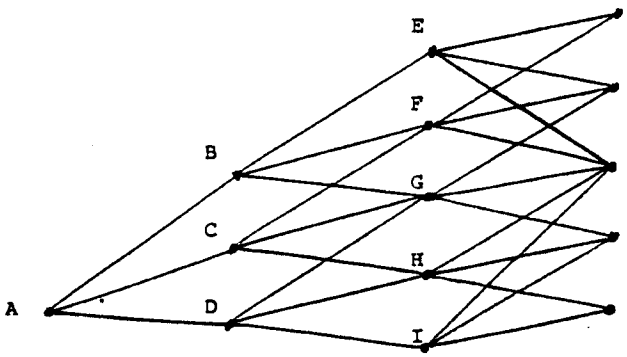
Exhibit 1 shows the tree that is produced when $x = r$, $a = 0.1$, $\sigma = 0.01$, Δt = one year, and the function $0.08 - 0.05e^{-0.18t}$ defines the t-year zero rate.[1] Note that the branching is non-standard at nodes E (where j is positive and large) and I (where j is large and negative).

II. MODELING TWO INTEREST RATES SIMULTANEOUSLY

Certain types of interest rate derivatives require

EXHIBIT 1

TREE ASSUMED FOR BOTH r_1 AND r_2 — CORRESPONDS TO THE PROCESS $dr = [\theta(t) - ar]dt + \sigma dz$ WHERE $a = 0.1$, $\sigma = 0.01$, Δt = ONE YEAR, AND THE t-YEAR ZERO RATE IS $0.08 - 0.05e^{-0.18t}$



Node	A	B	C	D	E	F	G	H	I
r	3.82%	6.93%	5.20%	3.47%	9.71%	7.98%	6.25%	4.52%	2.79%
p_u	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
p_m	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
p_d	0.167	0.222	0.167	0.122	0.087	0.222	0.167	0.122	0.887

the yield curves in two different countries to be modeled simultaneously. Examples are diff swaps and options on diff swaps. Here we explain how the procedure in Hull and White [1994] can be extended to accommodate two correlated interest rates.

For ease of exposition, we suppose the two countries are the United States and Germany, and that cash flows from the derivative under consideration are to be realized in U.S. dollars (USD). We first build a tree for the USD short rate and a tree for the deutschemark (DM) short rate using the procedure outlined above and described in more detail in Hull and White [1994]. As a result of the construction procedure, the USD tree describes the evolution of USD interest rates from the viewpoint of a risk-neutral USD investor, and the DM tree describes the evolution of DM rates from the viewpoint of a risk-neutral DM investor.

Adjusting the Deutschemark Tree

Since cash flows are realized in USD, the DM tree must be adjusted so that it reflects the evolution of rates from the viewpoint of a risk-neutral U.S. investor rather than a risk-neutral DM investor.

Define

r_1: USD short rate from the viewpoint of a risk-neutral U.S. investor;

r_2: DM short rate from the viewpoint of a risk-neutral DM investor;

r_2^*: DM short rate from the viewpoint of a risk-neutral U.S. investor;

X: the exchange rate (number of USD per DM);

σ_X: volatility of exchange rate, X;

ρ_X: instantaneous coefficient of correlation between the exchange rate, X, and the DM interest rate, r_2;

ρ: instantaneous coefficient of correlation between the interest rates, r_1 and r_2.

Suppose that the processes for r_1 and r_2 are

$$dx_1 = [\theta_1(t) - a_1x_1]dt + \sigma_1 dz_1$$

and

$$dx_2 = [\theta_2(t) - a_2x_2]dt + \sigma_2 dz_2$$

where $x_1 = f_1(r_1)$ and $x_2 = f_2(r_2)$ for some functions f_1 and f_2, and dz_1 and dz_2 are Wiener processes with correlation, ρ. The reversion rate parameters, a_1 and a_2, and the standard deviations, σ_1 and σ_2, are constant. The drift parameters, θ_1 and θ_2, are functions of time.

We show in Appendix A that the process for r_2^* is

$$dx_2^* = [\theta_2(t) - \rho_X\sigma_2\sigma_X - a_2x_2^*]dt + \sigma_2 dz_2$$

where $x_2^* = f_2(r_2^*)$. The effect of moving from a DM risk-neutral world to a USD risk-neutral world is to reduce the drift of x_2 by $\rho_X\sigma_2\sigma_X$. The expected value of x_2 at time t is reduced by

$$\int_0^t \rho_X\sigma_X\sigma_2 e^{-a_2(t-\tau)}\,d\tau = \frac{\rho_X\sigma_2\sigma_X}{a_2}(1 - e^{-a_2t})$$

To adjust the DM tree so that it reflects the viewpoint of a risk-neutral U.S. investor, the value of x_2 at nodes at time $i\Delta t$ (i.e., after i time steps) should therefore be reduced by[2]

$$\frac{\rho_X \sigma_2 \sigma_X}{a_2}(1 - e^{-a_2 i\Delta t})$$

Note that this is true for all functions f_2, not just $f_2(r) = r$.

To give an example, we suppose that $f_1(r_1) = r_1$; $f_2(r_2) = r_2$; $a_1 = a_2 = 0.1$; and $\sigma_1 = \sigma_2 = 0.01$. We also suppose that the t-year zero rate in both USD and DM is $0.08 - 0.05e^{-0.18t}$ initially. (This resembles the U.S. yield curve at the beginning of 1994.)

The tree initially constructed for r_1 and r_2 when $\Delta t = 1$ is shown in Exhibit 1. Suppose the correlation between r_2 and the exchange rate $\rho_X = 0.5$, and the exchange rate volatility $\sigma_X = 0.15$. To create the tree for r_2^* from the r_2 tree, the interest rates represented by nodes on the tree at the one-year point are reduced by $0.5 \times 0.01 \times 0.15\ (1 - e^{-0.1})/0.1 = 0.00071$ (0.071%). Similarly, interest rates at the two- and three-year points are reduced by 0.00136 and 0.00194, respectively.

Constructing the Tree Assuming Zero Correlation

We next combine the trees for USD and DM interest rates on the assumption of zero correlation. The result is a three-dimensional tree where nine branches emanate from each node. The probability associated with any one of the nine branches is the product of the unconditional probabilities associated with corresponding movements in the two short rates.

In the three-dimensional tree with the assumed parameter values, there is one node at time 0, nine nodes at time Δt, twenty-five nodes at time $2\Delta t$, twenty-five nodes at time $3\Delta t$, and so on. (Note that the effect of mean reversion is to curtail the rate at which the number of nodes on the tree increases.) Each node at time $i\Delta t$ on the combined tree corresponds to one node at time $i\Delta t$ on the first tree and one node at time $i\Delta t$ on the second tree. We will use a notation where node XY on the combined tree corresponds to node X on the first tree and node Y on the second tree.

The nine nodes at time Δt and the probability of reaching each of the nodes from the initial node AA are:

BD	BC	BB
0.0278	0.1111	0.0278
CD	CC	CB
0.1111	0.4444	0.1111
DD	DC	DB
0.0278	0.1111	0.0278

For example, the probability of reaching node BC from node AA is $0.1667 \times 0.6666 = 0.1111$. To illustrate how probabilities are calculated over the second time step, consider node BD, where $r_1 = 6.93\%$, $r_2 = 3.47\%$, and $r_2^* = 3.40\%$. From B we branch to E, F, G, and from D to G, H, I. So from BD we branch to the nine combinations of E, F, G and G, H, I. The nine nodes that can be reached from node BD together with their probabilities are:

EI	EH	EG
0.0149	0.0800	0.0271
FI	FH	FG
0.0800	0.4303	0.1456
GI	GH	GG
0.0271	0.1456	0.0493

For example, the probability of reaching node EG from node BD is $0.122 \times 0.222 = 0.0271$, the product of the probability of branching from B to E and from D to G.

To express the calculations more formally, we define node (i, j) as the node on the r_1 tree at time period $i\Delta t$ at which the number of prior up-moves in the interest rate minus the number of prior down-moves is j. Similarly, node (i, k) is the node on the r_2^* tree at time period $i\Delta t$ at which the number of prior up-moves in the interest rate minus the number of prior down-moves is k. Let $r_1(i, j)$ be the value of r_1 at node (i, j), and $r_2^*(i, k)$ be the value of r_2^* at node (i, k) on the r_2^* tree.

The three-dimensional tree is a combination of the two two-dimensional trees. At time $i\Delta t$ the nodes are denoted (i, j, k) where $r_1 = r_1(i, j)$ and $r_2^* = r_2^*(i, k)$ for all j and k. Assume the probabilities associat-

ed with the upper, middle, and lower branches emanating from node (i, j) in the r_1 tree are p_u, p_m, and p_d. Similarly, assume that the upper, middle, and lower branches emanating from node (i, k) in the r_2^* tree are q_u, q_m, and q_d.

In the three-dimensional tree there are nine branches emanating from the (i, j, k) node. The probabilities associated with the nine branches are:

		r_1-Move		
		Lower	Middle	Upper
	Upper	p_dq_u	p_mq_u	p_uq_u
r_2^*-Move	Middle	p_dq_m	p_mq_m	p_uq_m
	Lower	p_dq_d	p_mq_d	p_uq_d

Building in Correlation

We now move on to consider the situation where the correlation between r_1 and r_2^*, ρ, is nonzero. Suppose first that the correlation is positive. The geometry of the tree is exactly the same, but the probabilities are adjusted to be:

		r_1-Move		
		Lower	Middle	Upper
	Upper	$p_dq_u - \varepsilon$	$p_mq_u - 4\varepsilon$	$p_uq_u + 5\varepsilon$
r_2^*-Move	Middle	$p_dq_m - 4\varepsilon$	$p_mq_m + 8\varepsilon$	$p_uq_m - 4\varepsilon$
	Lower	$p_dq_d + 5\varepsilon$	$p_mq_d - 4\varepsilon$	$p_uq_d - \varepsilon$

Note that the sum of the adjustments in each row and column is zero. As a result, the adjustments do not change the mean and standard deviations of the unconditional movements in r_1 and r_2^*. The adjustments have the effect of inducing a correlation between r_1 and r_2^* of 36ε. The appropriate value of ε is therefore $\rho/36$.

The choice of the probability adjustments is motivated by the fact that in the limit as Δt tends to zero the probabilities tend to $p_u = q_u = 1/6$, $p_m = q_m = 2/3$, and $p_d = q_d = 1/6$. When the correlation is 1.0, the adjusted probability matrix is in the limit:

		r_1-Move		
		Lower	Middle	Upper
	Upper	0	0	1/6
r_2^*-Move	Middle	0	2/3	0
	Lower	1/6	0	0

For correlations between 0 and 1, the correlation matrix is the result of interpolating between the matrix for a correlation of 0 and the matrix for a correlation of 1.[3]

Suppose $\rho = 0.2$ so that $\varepsilon = 0.00556$. The probabilities on the branches emanating from node AA become:

BD	BC	BB
0.0222	0.0889	0.0556
CD	CC	CB
0.0889	0.4889	0.0889
DD	DC	DB
0.0556	0.0889	0.0222

while the probabilities on branches emanating from node BD become

EI	EH	EG
0.0093	0.0578	0.0549
FI	FH	FG
0.0578	0.4747	0.1234
GI	GH	GG
0.0549	0.1234	0.0437

For example, the probability of moving from node AA to node BC is $0.1111 - 4 \times 0.00556 = 0.0889$; the probability of moving from node BD to node EG is $0.0271 + 5 \times 0.00556 = 0.0549$.

For negative correlations, the procedure is the same, except that the probabilities are:

		r_1-Move		
		Lower	Middle	Upper
	Upper	$p_dq_u + 5\varepsilon$	$p_mq_u - 4\varepsilon$	$p_uq_u - \varepsilon$
r_2^*-Move	Middle	$p_dq_m - 4\varepsilon$	$p_mq_m + 8\varepsilon$	$p_uq_m - 4\varepsilon$
	Lower	$p_dq_d - \varepsilon$	$p_mq_d - 4\varepsilon$	$p_uq_d + 5\varepsilon$

In this case, $\varepsilon = -\rho/36$. For a correlation of -0.2, the probability on the branch from AA to BC is as before: $0.1111 - 4 \times 0.00556 = 0.0889$; the probability of moving from node BD to node EG is $0.0271 - 0.00556 = 0.0215$.

The only difficulty with this procedure is that some probabilities are liable to become negative at some nodes. We deal with this as follows. At any node where $\varepsilon = \rho/36$ leads to negative probabilities, we use the maximum value of ε for which probabilities are non-negative.

To illustrate this rule, suppose that $\rho = 0.8$ instead of 0.2 in our earlier example, so that the calculated value of ε is 0.02222. Using this value of ε at node BD would cause the probabilities on the branches to EH, EI, and FI to become negative. The maximum values of ε for which the three probabilities are non-negative are 0.0200, 0.0149, and 0.0200, respectively. We would therefore use a value of ε of 0.0149, which corresponds to $\rho = 0.5364$, for all branches emanating from BD.

As Δt approaches zero, p_u, p_m, and p_d approach 1/6, 2/3, and 1/6. Also q_u, q_m, and q_d approach 1/6, 2/3, and 1/6. The proportion of the probability space over which it is necessary to adjust ε reduces to zero. Although the procedure we have outlined does induce a small bias in the correlation, this bias disappears in the limit as Δt approaches zero. As a result the procedure converges.

An Example

We illustrate the procedure by using it to price a security whose payoff is calculated by observing the three-month DM rate in three years' time and applying it to a USD principal of 100. This is one component of a diff swap in which the swap payment in dollars is determined by the difference between the German and U.S. interest rates.

As before, we suppose that the process for both short rates is

$$dr = [\theta(t) - ar]dt + \sigma dz$$

with $a = 0.1$ and $\sigma = 0.01$.

The t-year zero-coupon bond rate in each country is assumed to be $0.08 - 0.05e^{-0.18t}$, and the volatility of the exchange rate is assumed to be 15% per year. This example enables us to test the convergence of the tree-building procedure since the security can be valued analytically using the formulas in Wei [1994].

Recall that ρ_X is defined as the correlation between the exchange rate and the DM rate, and ρ is the correlation between the two interest rates. Exhibit 2 shows results for a number of combinations of ρ and ρ_X. It illustrates that prices calculated by the tree do converge rapidly to the analytic price in a variety of different correlation environments.

EXHIBIT 2
VALUE OF A SECURITY WHOSE PAYOFF IS CALCULATED BY OBSERVING THE THREE-MONTH DM RATE IN THREE YEARS' TIME AND APPLYING IT TO A USD PRINCIPAL OF 100

Number of Time Steps	$\rho = 0.5$, $\rho_X = 0.5$	$\rho = 0.5$, $\rho_X = -0.5$	$\rho = -0.5$, $\rho_X = 0.5$	$\rho = -0.5$, $\rho_X = -0.5$
5	1.3971	1.4834	1.4045	1.4909
10	1.3976	1.4839	1.4055	1.4918
20	1.3978	1.4842	1.4059	1.4923
40	1.3980	1.4844	1.4062	1.4926
60	1.3981	1.4845	1.4063	1.4927
80	1.3981	1.4845	1.4063	1.4927
100	1.3981	1.4845	1.4064	1.4928
Analytic	1.3981	1.4845	1.4064	1.4928

Note: Both USD and DM rates are assumed to follow the process $dr = [\theta(t) - ar]dt + \sigma dz$ where $a = 0.1$, $\sigma = 0.01$, Δt = one year, and the t-year zero rate is $0.08 - 0.05e^{-0.18t}$.

III. TWO-FACTOR MODELS OF A SINGLE TERM STRUCTURE

Now we consider two-factor models of the form

$$df(r) = [\theta(t) + u - af(r)]dt + \sigma_1 dz_1 \qquad (3)$$

where the drift parameter u has an initial value of zero, is stochastic, and follows the process

$$du = -budt + \sigma_2 dz_2$$

As in the one-factor models considered in Hull and White [1994], the parameter $\theta(t)$ is chosen to make the model consistent with the initial term structure. The stochastic variable u is a component of the reversion level of r and itself reverts to a level of zero at rate b.[4] The parameters a, b, σ_1, and σ_2 are constants, and dz_1 and dz_2 are Wiener processes with

instantaneous correlation ρ.

This model provides a richer pattern of term structure movements and a richer pattern of volatility structures than the one-factor models considered at the outset. Exhibit 3 gives an example of forward rate standard deviations and spot rate standard deviations that are produced using the model when $f(r) = r$, $a = 3$, $b = 0.1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.0145$, and $\rho = 0.6$.[5] The volatilities that are observed in the cap market often have the "humped" shape shown in this plot.

When $f(r) = r$, the model is analytically tractable. As shown in Appendix B, the bond price in that case has the form

$$P(t, T) = A(t, T) \exp[-B(t, T)r - C(t, T)u]$$

The price, c, at time t of a European call option on a discount bond is given by

$$c = P(t, s)N(h) - XP(t, T)N(h - \sigma_p)$$

where T is the maturity of the option, s is the maturity of the bond, X is the strike price,

$$h = \frac{1}{\sigma_p}\log\frac{P(t, s)}{P(t, T)X} + \frac{\sigma_p}{2}$$

and σ_p is as given in Appendix B. Since this is a two-factor model, the decomposition approach in Jamshidian [1989] cannot be used to price options on coupon- bearing bonds.

Constructing the Tree

To construct a tree for the model in Equation (3), we simplify the notation by defining $x = f(r)$ so that

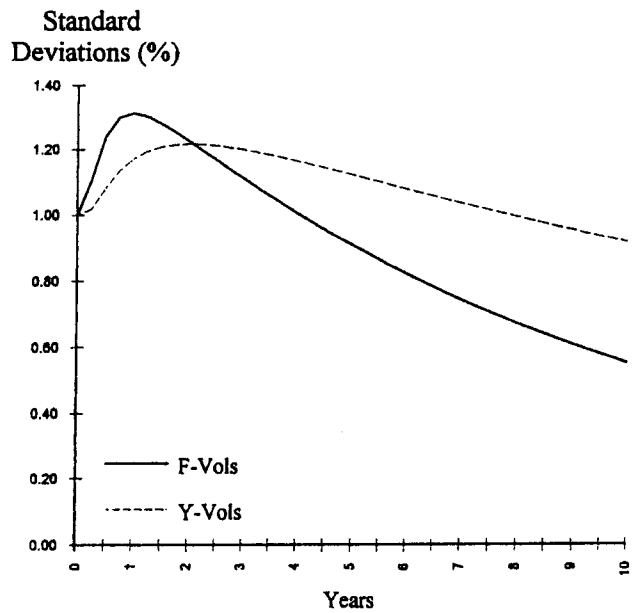
$$dx = [\theta(t) + u - ax]dt + \sigma_1 dz_1$$

with

$$du = -budt + \sigma_2 dz_2$$

Assuming $a \neq b$, we can eliminate the dependence of the first stochastic variable on the second by defining

EXHIBIT 3
FORWARD RATE VOLATILITIES (F-VOLS) AND SPOT RATE VOLATILITIES (Y-VOLS) IN TWO-FACTOR MODEL WHEN $f(r) = r$, $a = 3$, $b = 0.1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.0145$, AND $\rho = 0.6$



$$y = x + \frac{u}{b - a}$$

so that

$$dy = [\theta(t) - ay]dt + \sigma_3 dz_3$$

$$du = -budt + \sigma_2 dz_2$$

where

$$\sigma_3^2 = \sigma_1^2 + \frac{\sigma_2^2}{(b - a)^2} + \frac{2\rho\sigma_1\sigma_2}{b - a}$$

and dz_3 is a Wiener process. The correlation between dz_2 and dz_3 is

$$\frac{\rho\sigma_1 + \sigma_2/(b - a)}{\sigma_3}$$

The approach described in Section II can be used to construct a tree for y and u on the assumption that $\theta(t) = 0$ and the initial values of y and u are zero. Using a similar approach to that described in Hull and White [1994], we can then construct a new tree by increasing the values of y at time $i\Delta t$ by α_i. The α_is are calculated using a forward induction technique similar to that described in Hull and White [1994] so that the initial term structure is matched. The details are given in Appendix C.

An Example

Exhibit 4 shows the results of using the tree to price three-month options on a ten-year discount bond. The parameters used are those that give rise to the humped volatility curve in Exhibit 3. The initial t-year zero rate is assumed to be $0.08 - 0.05e^{-0.18t}$ Five different strike prices are considered.

Exhibit 4 illustrates that prices calculated from the tree converge rapidly to the analytic price. The numbers here and in other similar tests we have carried out provide support for the correctness of the somewhat complicated formulas for A(t, T) and σ_P in Appendix B.

IV. CONCLUSIONS

This article shows that the approach in Hull and White [1994] can be extended in two ways. It can be used to model two correlated interest rates when each follows a process chosen from the family of one-factor models considered in our earlier article. It can also be used to implement a range of different two-factor models.

An interesting by-product of the research described here is a method for combining trinomial trees for two correlated variables into a single three-dimensional tree describing the joint evolution of the variables. This method can be extended so that it accommodates a range of binomial as well as trinomial trees.

EXHIBIT 4
VALUE OF THREE-MONTH OPTION ON A TEN-YEAR DISCOUNT BOND WITH A PRINCIPAL OF 100 FOR MODEL IN EXHIBIT 3

Number of	Strike Price*				
Time Steps	0.96	0.98	1.00	1.02	1.04
5	1.9464	1.0077	0.2890	0.0397	0.0012
10	1.9499	1.0077	0.2968	0.0370	0.0016
20	1.9512	1.0085	0.3000	0.0366	0.0015
40	1.9536	1.0107	0.3022	0.0369	0.0014
60	1.9529	1.0099	0.3020	0.0367	0.0014
Analytic	1.9532	1.0101	0.3023	0.0367	0.0014

*The strike price is expressed as a proportion of the forward bond price.

Note: The t-year zero rate is $0.08 - 0.05e^{-0.18t}$.

APPENDIX A

In this appendix we show how to calculate the process for the DM interest rate from the viewpoint of a USD investor. Our approach is an alternative to that in Wei [1994].

Define Z as the value of a variable seen from the perspective of a risk-neutral DM investor and Z^* as the value of the same variable seen from the perspective of a risk-neutral USD investor. Suppose that Z depends only on the DM risk-free rate so that

$$dZ = \mu(Z)Zdt + \sigma(Z)Z\, dz_2$$

where dz_2 is the Wiener process driving the DM risk-free rate, and μ and σ are functions of Z. The work of Cox, Ingersoll, and Ross [1985] and others shows that the process for Z^* has the form:

$$dZ^* = [\mu(Z^*) - \lambda\sigma(Z^*)]Z^*dt + \sigma(Z^*)Z^*dz_2$$

where the risk premium, λ, is a function of Z^*.[6]

We first apply this result to the case where the variable under consideration is the DM price of a DM discount bond. Define P as the value of this variable from the viewpoint of a risk-neutral DM investor and P^* as its value from the viewpoint of a risk-neutral USD investor. From the perspective of a risk-neutral DM investor, the variable is the price of a traded security so that:

$$dP = r_2Pdt + \sigma_PPdz_2 \qquad \text{(A-1)}$$

where σ_P is the volatility of P, and other variables are as defined in the body of the article. Hence:[7]

$$dP^* = [r_2^* - \lambda\sigma_P]P^*dt + \sigma_PP^*dz_2 \qquad \text{(A-2)}$$

The risk-neutral process for the exchange rate, X, from the viewpoint of a risk-neutral USD investor is

$$dX = (r_1 - r_2^*)Xdt + \sigma_X X dz_X \qquad \text{(A-3)}$$

where dz_X is a Wiener process.

The variable XP^* is the price in USD of the DM bond. The drift of XP^* in a risk-neutral USD world must therefore be r_1XP^*. From Equations (A-2) and (A-3), this drift can also be written as

$$XP^*(r_1 - \lambda\sigma_P + \rho_X\sigma_X\sigma_P)$$

It follows that

$$\lambda = \rho_X\sigma_X$$

When moving from a risk-neutral DM investor to a risk-neutral USD investor there is a market price of risk adjustment of $\rho_X\sigma_X$.

We can now apply the general result given for Z at the beginning of Appendix A to the variable $f(r_2)$. We are assuming that:

$$df_2(r_2) = [\theta_2(t) - af_2(r_2)]dt + \sigma_2 dz_2$$

Hence

$$df_2(r_2^*) = [\theta_2(t) - \rho_X\sigma_X\sigma_2 - af_2(r_2^*)]dt + \sigma_2 dz_2$$

APPENDIX B

In this appendix we show that the complete yield curve can be calculated analytically from r and u in the model considered in Section III when $f(r) = r$.

The differential equation satisfied by a discount bond price, f, is

$$f_t + [\theta(t) + u - ar]f_r - buf_u + \frac{1}{2}\sigma_1^2 f_{rr} + \frac{1}{2}\sigma_2^2 f_{uu} + \rho\sigma_1\sigma_2 f_{ru} - rf = 0$$

By direct substitution, a solution to this equation is

$$f = A(t, T)e^{-B(t,T)r - C(t,T)u}$$

providing

$$B_t - aB + 1 = 0 \qquad \text{(B-1)}$$

$$C_t - bC + B = 0 \qquad \text{(B-2)}$$

$$A_t - \theta(t)AB + \frac{1}{2}\sigma_1^2 AB^2 + \frac{1}{2}\sigma_2^2 AC^2 + \rho\sigma_1\sigma_2 ABC = 0 \qquad \text{(B-3)}$$

The solution to Equation (B-1) that satisfies the boundary condition $B(t, t) = 0$ is

$$B(t, T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

The solution to Equation (B-2) that is consistent with this and satisfies the boundary condition $C(t, t) = 0$ is

$$C(t, T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

By direct substitution, the solution to Equation (B-3) for A and θ that satisfies the boundary conditions $A(t, T) = A(0, T)$ when $t = 0$ and $A(t, T) = 1$ when $t = T$ is

$$\log A(t, T) = \log\frac{P(0, T)}{P(0, t)} + B(t, T)F(0, T) + \phi(0, t)B(t, T) + \int_0^t \phi(0, \tau)d\tau - \int_0^t \phi(\tau, T)d\tau$$

$$\theta(t) = F_t(0, t) + aF(0, t) + \phi_t(0, t) + a\phi(0, t)$$

where $F(t, T)$ is the instantaneous forward rate for time T as seen at time t, and

$$\phi(t, T) = \frac{1}{2}\sigma_1^2 B(t, T)^2 + \frac{1}{2}\sigma_2^2 C(t, T)^2 + \rho\sigma_1\sigma_2 B(t, T)C(t, T)$$

This reduces to

$$\log A(t, T) = \log\frac{P(0, T)}{P(0, t)} + B(t, T)F(0, t) - \eta$$

where

$$\eta = \frac{\sigma_1^2}{4a}(1 - e^{-2at})B(t, T)^2 - \rho\sigma_1\sigma_2[B(0, t) \times C(0, t)B(t, T) + \gamma_4 - \gamma_2] - \frac{1}{2}\sigma_2^2[C(0, t)^2 \times B(t, T) + \gamma_6 - \gamma_5]$$

$$\gamma_1 = \frac{e^{-(a+b)T}[e^{(a+b)t} - 1]}{(a + b)(a - b)} - \frac{e^{-2aT}(e^{2at} - 1)}{2a(a - b)}$$

$$\gamma_2 = \frac{1}{ab}\left[\gamma_1 + C(t, T) - C(0, T) + \frac{1}{2}B(t, T)^2 - \frac{1}{2}B(0, T)^2 + \frac{t}{a} - \frac{e^{-a(T-t)} - e^{-aT}}{a^2}\right]$$

$$\gamma_3 = -\frac{e^{-(a+b)t} - 1}{(a - b)(a + b)} + \frac{e^{-2at} - 1}{2a(a - b)}$$

$$\gamma_4 = \frac{1}{ab}\left[\gamma_3 - C(0, t) - \frac{1}{2}B(0, t)^2 + \frac{t}{a} + \frac{e^{-at} - 1}{a^2}\right]$$

$$\gamma_5 = \frac{1}{b}\left[\frac{1}{2}C(t, T)^2 - \frac{1}{2}C(0, T)^2 + \gamma_2\right]$$

$$\gamma_6 = \frac{1}{b}\left[\gamma_4 - \frac{1}{2}C(0, t)^2\right]$$

The volatility function σ_P is given by

$$\sigma_P^2 = \int_0^t \{\sigma_1^2[B(\tau, T) - B(\tau, t)]^2 + \sigma_2^2[C(\tau, T) - C(\tau, t)]^2 + 2\rho\sigma_1\sigma_2[B(\tau, T) - B(\tau, t)] \times [C(\tau, T) - C(\tau, t)]\}d\tau$$

This shows that σ_P^2 has three components. Define

$$U = \frac{1}{a(a - b)}[e^{-aT} - e^{-at}]$$

and

$$V = \frac{1}{b(a - b)}[e^{-bT} - e^{-bt}]$$

The first component of σ_P^2 is

$$\frac{\sigma_1^2}{2a}B(t, T)^2(1 - e^{-2at})$$

The second is

$$\sigma_2^2\left[\frac{U^2}{2a}(e^{2at} - 1) + \frac{V^2}{2b}(e^{2bt} - 1) - 2\frac{UV}{a + b}(e^{(a+b)t} - 1)\right]$$

The third is

$$\frac{2\rho\sigma_1\sigma_2}{a}(e^{-at} - e^{-aT})\left[\frac{U}{2a}(e^{2at} - 1) - \frac{V}{a + b}(e^{(a+b)t} - 1)\right]$$

APPENDIX C

In this appendix we explain how the tree for the two-factor model discussed in Section III is fitted to the initial term structure. We assume that a three-dimensional tree for y and u has been constructed on the assumption that $\theta(t) = 0$

using the procedure in Section II, that the spacing between y-values on the tree is Δy, and that the spacing between the u-values on the tree is Δu. From Equation (4) when $y = j\Delta y$ and $u = k\Delta u$, the short rate, r is given by the initial tree to be

$$r = g\left[j\Delta y - \frac{k\Delta u}{b - a}\right]$$

where g is the inverse function of f.

As in Hull and White [1994], we work forward through the tree calculating the αs that must be added to the ys so that the tree is perfectly consistent with the initial term structure. Once the appropriate α for time $i\Delta t$, α_i, has been calculated, it is used to calculate Arrow-Debreu prices for the nodes at time $i\Delta t$. (The Arrow-Debreu price for a node is the present value of a security that pays off \$1 if the node is reached and zero otherwise.) These are then used to calculate α_{i+1}, and so on.

Define $Q_{i,j,k}$ as the present value of a security that pays off \$1 if $y = \alpha_i + j\Delta y$ and $u = k\Delta u$ at time $i\Delta t$, and zero otherwise. $Q_{0,0,0} = 1$. Similarly to Equation (6) in Hull and White [1994],

$$P_{m+1} = \sum_{j,k} Q_{m,j,k} \exp\{-g[\alpha_m + j\Delta y - k\Delta u/(b - a)]\Delta t\}$$

where the summation is taken over all values of j and k at time $m\Delta t$. When $f(r) = r$ so that $g(r) = r$ this can be solved analytically for α_m:

$$\alpha_m = \frac{\log \sum_{j,k} Q_{m,j,k} \exp\{j\Delta y - k\Delta u/(b - a)\} - \log P_{m+1}}{\Delta t}$$

In other situations, a one-dimensional Newton-Raphson search is required.

The Qs are updated as the tree is constructed using

$$Q_{m+1,j,k} = \sum_{j^*,k^*} Q_{m,j^*,k^*}\, q(j, k, j^*, k^*) \times \exp\{-g[\alpha_m + j^*\Delta y - k^*\Delta u/(b - a)]\Delta t\}$$

where in the summation j^* and k^* are set equal to all possible pairs of values of j and k at time $m\Delta t$. The variable $q(j, k, j^*, k^*)$ is the probability of a transition from node (m, j^*, k^*) to $(m + 1, j, k)$.

ENDNOTES

[1]Exhibit 1 is the same as Exhibit 3 in Hull and White [1994]. Fuller details of how it is developed are given in that article.

[2]Using the notation in Hull and White [1994], this is the amount by which α_i should be adjusted once the tree has been constructed. Since the r on the tree is the Δt-period rate rather than the instantaneous rate, the adjustment is exact only in the limit as Δt tends to zero.

[3]The procedure described here can (with appropriate modifications) be used to construct a three-dimensional tree for any two correlated variables from two trees that describe the movements of each variable separately. The two trees can be binomial or trinomial.

[4]There is no loss of generality in assuming that the reversion level of u is zero and that its initial value is zero. For example, if u reverts to some level c, $u^* = u - ct$ reverts to 0. We can define u^* as the second factor and absorb the difference between u and u^* in $\theta(t)$.

[5]These parameters are chosen to produce the desired pattern. Many other different volatility patterns can be achieved.

[6]λ is the difference between the market price of the DM interest rate risk from the perspective of a risk-neutral DM investor and the market price of DM interest rate risk from the perspective of a risk-neutral USD investor.

[7]Note that r_2 in Equation (A-1) becomes r_2^* in Equation (A-2). This is because the r_2 in (A-2) is, strictly speaking, $r_2(P)$. The Cox, Ingersoll, and Ross result outlined at the beginning of the appendix shows that this becomes $r_2(P^*)$ or r_2^* in Equation (A-2).

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ERRATUM

The first part of this article, "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models," published in the Fall 1994 issue of *The Journal of Derivatives*, contained an equation that was stated incorrectly.

The last equation on page 13 (unnumbered) should read

$$\alpha_m = \frac{\log\left(\sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta r \Delta t}\right) - \log P(0,\ m+1)}{\Delta t}$$