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Modeling the Short Rate: The Real and Risk-Neutral Worlds

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Abstract

Traditionally, derivatives researchers have tended to focus on the (risk-neutral) Q-measure because of its role in pricing. Since the crisis, risk management has assumed an increased importance and there is now a realization among many quants that the (real-world) P-measure should be given more attention. In this paper, we propose a way to construct a single forward-looking model for interest rates, which represents their evolution under both the Q-measure and P-measure (a joint measure model). As is well known, the market prices of contingent claims are independent of investor risk preferences. This means that risk preferences, and therefore real world processes, cannot be obtained from market prices alone. In this paper we present a simple way in which historical data can be used in conjunction with market prices to create a joint measure model for the short rate. The approach can be used for a wide range of interest rate models.

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Banks need to calculate a number of different measures associated with credit risk. The credit value adjustment (CVA) for the portfolio of derivatives with a counterparty is an adjustment to the value of the portfolio to reflect the possibility of a default by the counterparty. The exposure at default (EAD) measure is used in Basel II to calculate regulatory capital under the Advanced IRB approach. Potential future exposure (PFE), which is a high quantile of the exposure distribution at a future time, is often used by banks internally to set limits to counterparty credit risks.

All of these measures are calculated by carrying out a computationally intensive Monte Carlo simulation of relevant underlying market variables. To calculate CVA, the market variables should be simulated under the risk-neutral Q-measure. For EAD and PFE, the real world P-measure should be used. Because of the difficulties of estimating real world processes, EAD and PFE are sometimes calculated from the same risk neutral Monte Carlo simulation as the one used to calculate CVA. This can only be justified for the narrow purpose of gaining more insight into where CVA is coming from or because no approach is available for estimating real world processes.

Risk neutral processes can usually be estimated fairly easily from market prices, but it is more difficult to know real world processes. The problem of deriving real world processes from risk-neutral processes is not confined to credit risk modeling. For example, liquidity risk managers require a real-world Monte Carlo simulation to determine potential future funding problems; fund managers need to carry out real-world simulations of market variables to evaluate investment strategies.

Regulators understand the problems of using real world measures. In Basel II when discussing EAD they note (footnote 240): “In theory, the expectations should be taken with respect to the actual probability distribution of future exposure and not the risk-neutral one. Supervisors recognise that practical considerations may make it more feasible to use the risk-neutral one. As a result, supervisors will not mandate which kind of forecasting distribution to employ”.

Obtaining a reasonable estimate of the real-world drift is less important for short-term risk exposures than for long-term risk exposures. This is because volatility has a much bigger effect than the drift in the short term. However, risk-related measures are often concerned with relatively long-term exposures. This is particularly true for pension funds and insurance companies. The choice of measure can also be important for the one-year horizon used in several types of regulatory capital calculations. Indeed, approximating the real-world drift with the risk-neutral drift is questionable for all but very short term horizons.

When we move from the real world to the risk-neutral world, or vice versa, Girsanov's theorem shows that the drifts of variables change but their volatilities remain the same. One approach to estimating the market price of risk is to use the capital asset pricing model and base the market price of risk of a variable on an estimate of its correlation with a market index such as S&P 500 (i.e., its systematic risk). This works reasonably well for some, but not all variables. In this paper we focus on interest rates and develop a more robust approach. Interest rates exhibit the opposite behavior to equity prices. When we move from the risk neutral world to the real world, the drift of an interest decreases rather than increases.

Our objective is to show how a plausible model of term structure movements under both real-world and risk-neutral measures (a joint measure model) can be constructed. Such a model can be used for both pricing and the determination of regulatory capital and internal risk measures. For the last 35 years, derivatives researchers have tended to focus on the (risk-neutral) Q-measure because of its role in pricing. With the post-crisis focus on risk management, there is now a realization among many quants that the (real-world) P-measure should be given more attention.

A risk-neutral interest rate model can be derived by choosing a functional form for the model and calibrating its parameters to market prices. Real-world models are more difficult to calibrate. They tend to rely on historical data and are therefore to some extent backward-looking. In this paper, we propose a way to construct a single forward looking model representing both Q-measure and P-measure processes (a joint measure model). As is well known, the market prices of contingent claims are independent of investor risk preferences. This means that risk preferences, and therefore real world processes, cannot be obtained from market prices alone. We propose a simple way in which historical data can be used in conjunction with market prices to create a joint measure model for the short rate. The approach can be used for a wide range of interest rate models.

Prior research concerned with real-world measures includes Stein (2013), who proposes ways in which the computational burden of carrying out Monte Carlo simulation under two measures can be eased and Ross (2011) who shows how the real world process for an equity index can be extracted from the risk neutral process and option prices when certain assumptions are made.

Background

We illustrate our approach with a single-factor Markov model for the short rate, *r*. Under the risk neutral measure, the most general form of the model is:

$$dr = \mu(t,r)dt + \sigma(t,r)dz \tag{1}$$

where the parameters μ and σ are deterministic functions of *r* and *t*, and *dz* is a Wiener process.

Rebonato (1998) provides a good description of alternative models and the role of the market price of interest rate risk. The real-world model corresponding to equation (1) is

$$dr = \left(\mu\left(t,r\right) + \lambda(r,t,...)\sigma\left(t,r\right)\right)dt + \sigma\left(t,r\right)dz \tag{2}$$

where λ is the market price of interest rate risk. In the most general version of the model λ may depend on *r*, *t*, and other market variables. However, if λ depends on other market variables, the joint-measure model is no longer a single-factor model. If investors are risk averse, the drift of bond prices is higher in the real world than in the risk-neutral world. The drift of all interest rates including the short rate *r* is therefore lower in the real world than in the risk-neutral world so that λ is negative.

The volatility term $\sigma(t,r)$ is the same in equations (1) and (2). This is consistent with Girsanov's theorem. It is well known that in practice risk-neutral market implied volatilities differ from real-world historical volatilities. This is not a violation of Girsanov's theorem but simply a reflection of the fact that market-implied volatilities are forward-looking estimates while the historical estimates are backward-looking. Using historical estimates of realized volatilities is often seen as the only available choice in calibrating a model in real world measure. We believe that forward-looking estimates of the real-world process for *r* which incorporate market implied volatilities, or historical averages of market implied volatilities, should be included in the arsenal of available calibration methods for PFE, regulatory capital, fund management, and liquidity calculations. We will therefore describe how one can incorporate market implied data into the estimation of drift and volatility even when using the model for the purposes of computing real-world quantiles.

Estimating the Historical Market Price of Risk

A number of researchers have attempted to estimate the average market price of interest rate risk over periods of time in the past. For example, Stanton (1997), Cox and Pedersen (1999), and Ahmad and Wilmott (2007) estimate a value for λ that is consistent with the average slope of the term structure for maturities between zero and six months.[1] Stanton and Cox and Pedersen produce estimates between −0.5 and −2.5 based on US interest rate data for the period 1965 to 1995. Wilmott produces an estimate of –1.2 based on US interest rate data for the period 1982 to 2006.

These estimates of the market price of interest rate risk are surprisingly large. The average standard deviation calculated from short-term US interest rates over the last 50 years has been about 1.0%. If this is combined with a market price of risk of, say, –1.0 in equation (2), we find that the risk neutral and real world rates must diverge by about 1% in absolute terms over one year. As we will see in a moment, if taken at face value, this rate of divergence leads to implausible conclusions about the real world probability distribution of the interest rates at the 30 year horizon.

The exact magnitude of the divergence depends on $\mu(t, r)$ and $\sigma(t, r)$. Consider Vasicek's (1977) model where the drift process is a simple mean reverting function $\mu(t, r) = a(b - r)$ and σ is constant. The expected real interest rate in T years' time will be lower than the expected risk-neutral rate by $\left(\lambda\sigma\left(1-\exp\left(-aT\right)\right)/a\right)$. When the reversion rate, a, is 0.1 and T = 30 years the

[1] Note that Stanton provides an estimate for λ times the standard deviation of the short rate rather than λ itself.

difference is about 9.50% in absolute terms. When the reversion rate is 0.0 and T = 30 years the difference is 30% in absolute terms.[2]

For term structure models, such as Black and Karasinski (1991), in which $\sigma(t, r)$ approaches zero as r approaches zero, if the market price of risk equals −1.0 the expected future real-world rate in 30 years is unrealistically close to zero. For term structure models where $\sigma(0, t)$ is not zero then the drift change usually results in the expected future real-world rate being negative.

This puzzling characteristic of the market price of interest rate risk is also apparent in empirical analyses of interest-rate data. A simple approximation that can be used to estimate the market price of risk is based on the historical average return earned by bonds of different maturities. Suppose that the continuously compounded long-run average short-term interest rate is r_0 and the long run average instantaneous futures rate for maturity T is $F(T)$.[3] If the interest rate process is stationary these long run average rates are estimates of the expected rates. Since the expected future interest rate in a risk-neutral world equals the futures rate, moving from the real world to the risk-neutral world increases the expected rate at time T by $F(T)-r_0$. Assuming a constant σ in equations (1) and (2) and a μ that is independent of r (so that there is no mean reversion), it follows that $\lambda\sigma T = r_0 - F(T)$ so that

$$\lambda_T = -\frac{F(T)-r_0}{\sigma T} \tag{3}$$

The subscript on λ indicates the rate maturity that the estimate is based on.

When there is mean reversion two things happen. First, the convexity adjustment used to convert forward prices to futures prices is lower so that the estimate of the futures rate, $F(T)$, is lower. This lowers the market price of risk. Second, for a particular value of $F(T)$ a higher market price of risk is necessary to increase the expected rate by $F(T)-r_0$. This increases the estimate of the market price of risk.

[2] The same is true of the Hull and White (1990) extension of the Vasicek model

[3] The long run average futures rate is calculated from the long run average forward rate by making a convexity adjustment.

We used the approach just outlined to provide a rough estimate the average historical market price of risk from daily US interest rate data for the period from January 1982 to January 2014 with three different estimates of the reversion rates (0%, 5%, and 10%). Our results are based on the Hull and White (1990) model. (This is a convenient model to use because of its analytic properties, but tests indicate that similar results are obtained from other one-factor models.) The estimates were based on the long-run average of rates over the whole period as well as for two sub-periods: January 1982 to December 1997 and January 1998 to January 2014. The estimate of σ was based on the standard deviation of daily rate changes in the appropriate period. The two sub-periods differ dramatically in that rates and rate volatilities are much higher in the first than in the second. The estimated values of λ_T for the three periods considered and a reversion rate of 0.05 are shown in Figure 1. The estimated values for the whole period using the three different reversion rates are shown in Figure 2.

Figure 1 shows that the results are similar for all three periods. The estimated market prices of risk based on the 3-month and 6-month rates are: –1.0 for the full sample, –1.1 for the 1982 to 1997 subset and –1.0 for the 1998 to 2014 sample. These estimates are consistent with the estimates of other researchers mentioned earlier. However, the estimates based on longer-term rates are much lower. With a reversion rate of 0.05, the estimate for the 30-year maturity is −0.19. With a reversion rate of 0.10, the estimate is −0.21.

How should we interpret the fact that these estimates depend on the maturity of the rates used? Interest rate movements depend on multiple factors. This means that the λ estimated from a one-factor model may depend on the instrument maturity from which it is estimated. It is possible that the observed dependence may be explained by a two-factor model where the first factor mainly affects the short rate and has a high market price of risk while the second factor mainly affects long rates and has a low market price of risk.

In what follows we show how the shortcomings of estimating the market price of risk from a one-factor short rate model can be overcome by making the market price of risk a function of time. This is analogous to the way analysts use functions of time in one-factor short rate models to fit the current term structure of interest rates and (sometimes) the term structure of volatilities.

Constructing the Joint Measure Model

The approach we will describe can be implemented for a wide range of one-factor term structure models. We will use a mean-reverting version of a model proposed by Deguillaume *et al*. (2013), which was found to fit historical interest rate data very well and which was found to fit cap prices well by Hull and White (2014). Specifically, we will assume that the short rate follows the process

$$dr = \left[\theta(t) - ar\right]dt + \sigma(r)dz$$

where

$$\sigma(r) = \begin{cases} sr/0.015 \quad \text{when } r \le 0.015 \\ s \text{ when } 0.015 < r < 0.06 \\ sr/0.06 \text{ when } r \ge 0.06 \end{cases}$$

In this model the volatility is lognormal for rates less than 1.5%, normal for rates between 1.5% and 6%, and lognormal for rates greater than 6%.[4] We assume that the mean reversion parameter, *a*, is 0.05. The model therefore has one free parameter, *s*.

We first estimate the historical average term structure using data going back to 1982. We then estimate the average value of *s* experienced since 1982. This was about 1.05%.[5] We then used the procedure described in Hull and White (2014) to construct a tree describing the average historical behavior of the term structure.

Suppose that the value of the market price of risk at time step *i* is λ_i and that the drift of the short rate at node *j* is μ_{ij}. In the real world, this drift becomes $\mu_{ij} + \lambda_i \sigma_{ij}$ where σ_{ij} is the standard deviation of the short rate at the *j*th node at the *i*th time step. Now, the average drift of the short rate at each time can reasonably be assumed to be zero in the real world. This means that, when we move from the risk-neutral world to the real world, the mean value of the short rate at all times should be equal to the historical average short rate, i.e., the root of the historical average

[4] In Deguillaume *et al* (2013), the break points are not necessarily exactly 1.5% and 6% and the slope of the term structure beyond the second break point can be different from our assumption.
[5] We assume that a similar estimate would have been obtained if historical market data had been used to imply a historical average *s*; i.e., we assume that historical and implied volatilities are the same on average.

term structure. For our data, the historical average short rate was 4.4%. We therefore advance through the tree searching at time step *i* for the value of the market price of risk, $\lambda_{i,}$, that reduces the expected short rate to 4.4%.

We emphasize that the λ_i are not a true market prices of risk. We refer to them as the local prices of risk. If the term structure really were driven by a one-factor model, then we would find that the local price of risk was the same at all times and equal to the market price of risk. In reality, the time varying λ are nothing more than a "quick-and-dirty" way of fitting a one-factor model to the data.

The next step is to calibrate the model to current market data. We used market data on cap prices and term structure data for December 2, 2003, and constructed a tree for the short rate in a similar way to that in Hull and White (2014).[6] The value of *s* was about 2.1%.We then used the estimated λ_i to reduce the drift at each node of the tree. This produces a tree for the real world evolution of the short rate.

This tree is based on the assumption that the local prices of risk are constant through time. It can be adjusted to ensure that a target expected short rate is matched at a distant future time. The target future rate could be the average short rate observed in the past (4.4% for our data). This is reasonable if the process followed by interest rates is stationary because observations from the past can be considered to be random samples from the short-rate probability distribution. But if future monetary and fiscal policies are expected to be materially different from those in the past, the approach is questionable. (However, any other approach based on historical data will then also be questionable.) The user can then provide another estimate (e.g., one based on personal belief or a macroeconomic model) for a future expected short rate.

The adjustment (admittedly fairly ad hoc) is simply to scale the λ_i by a constant parameter so that the target long-run average short rate is reached. The maturity that is used to define the long-run average short rate is a matter of judgment. Because of the flatness of the term structure at long maturities we find that it does not matter whether the long-run rate is assumed to be that observed in 15, 20, 25, or 30 years.

[6] The model here is a simplification of the model in Hull and White (2014) and so estimated parameters are slightly different.

The decision on whether to make the adjustment depends on whether one believes that it is the market price of risk or long run average level of the short rate that does not change over time. If it is believed that the market price of risk is constant then no adjustment should be made.

Figures 3, 4, and 5 show the probability distribution of the future short rate in 1, 5, and 30 years on December 2, 2013. As already described, the risk-neutral short rate distributions are created from a tree calibrated to cap prices and the term structure. The real-world rate distribution is created by using the same tree but adjusting the drift by the λ_i. (To produce these figures we did not adjust to match a target for the long-run average short rate.) It can be seen that, even one year in the future, there is substantial difference between the real-world and risk-neutral distribution of interest rates. The real-world short rate distribution is tightly clustered around the current short rate while much more variation is seen in the risk-neutral rate. The distributions diverge as we look further ahead, but low rates are always much more likely under the distribution of the real rate.

Figures 6 and 7 show how the mean and standard deviation of the real and risk-neutral short rate distributions increase as the time horizon is increased for both the situation where no adjustments are made to match a long-run target and situations where these adjustments are made. In the former case, the long run expected real interest rate is about 2.9%; in the latter case it is 4.4%.

Conclusions

This paper provides a first attempt at constructing and calibrating a single market-implied model for both the Q-measure and P-measure distributions over long time horizons (the joint measure model). We illustrate our procedure for a simple one-factor short rate model. With minor modifications, the same approach could be applied to other interest rate models. We hope that this paper will stimulate further research into real world measure models for other asset classes and for multifactor real world models for interest rates.

When a tree-building procedure has a fixed lattice of short rates, as Hull and White (2014) does, a single tree can be used for both P- and Q-measures. A key point is that a move from the Q-measure to the P-measure does not change the value of a portfolio at each node. Only the probabilities on each branch of the tree change. Once the two sets of probabilities have been determined in the way described in this paper, the same tree can be correctly used for both valuation and scenario analysis.

The real-world probability distributions are market-implied in the sense that they change as the market conditions change. The model permits the calculation of PFE and similar measures using the same market inputs as those used in the calculation of CVA combined with historical data on the term structure. If desired, a macroeconomic forecast for long-run average short rates can be used.

Finally, this paper shows that there is a dependence of the market price of interest rate risk on the maturity of interest rates used for its estimation. This is a phenomenon which, as far as we know, has not be commented on in the finance and economics literature. Further research is necessary to fully analyze the nature and sources of this dependence. Our proposed calibration procedure can be extended to multifactor short-rate models and models in forward measure, such as two- or three-factor LIBOR market models and SABR models.[7] In a multifactor model, each factor has its own market price of risk. A simple assumption would be that λ is the same for all factors. However, a factor that primarily affects short-term interest rates may have a higher λ than a factor that primarily affects long-term rates. Using a multifactor model, it may therefore be possible to explain the term structure of the local price of risk.

[7] See Rebonato (2009) for a description of these models.

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Figure 1

Dependence of the Estimated Market Price of Risk on the Maturity of the Bonds Used in the Estimation for Different Time Periods. Reversion rate = 5%.

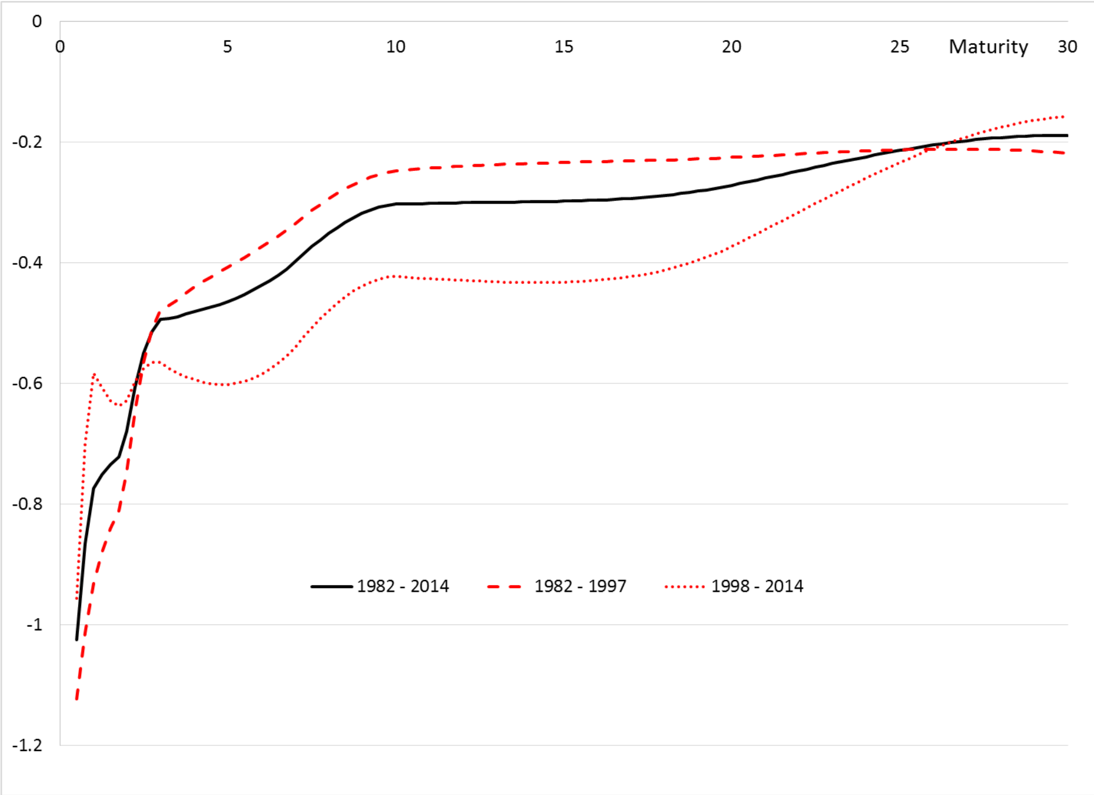

Figure 2

Dependence of the Estimated Market Price of Risk on the Maturity of the Bonds Used in the Estimation Using Different Reversion Rates and Data from 1982 to 2014

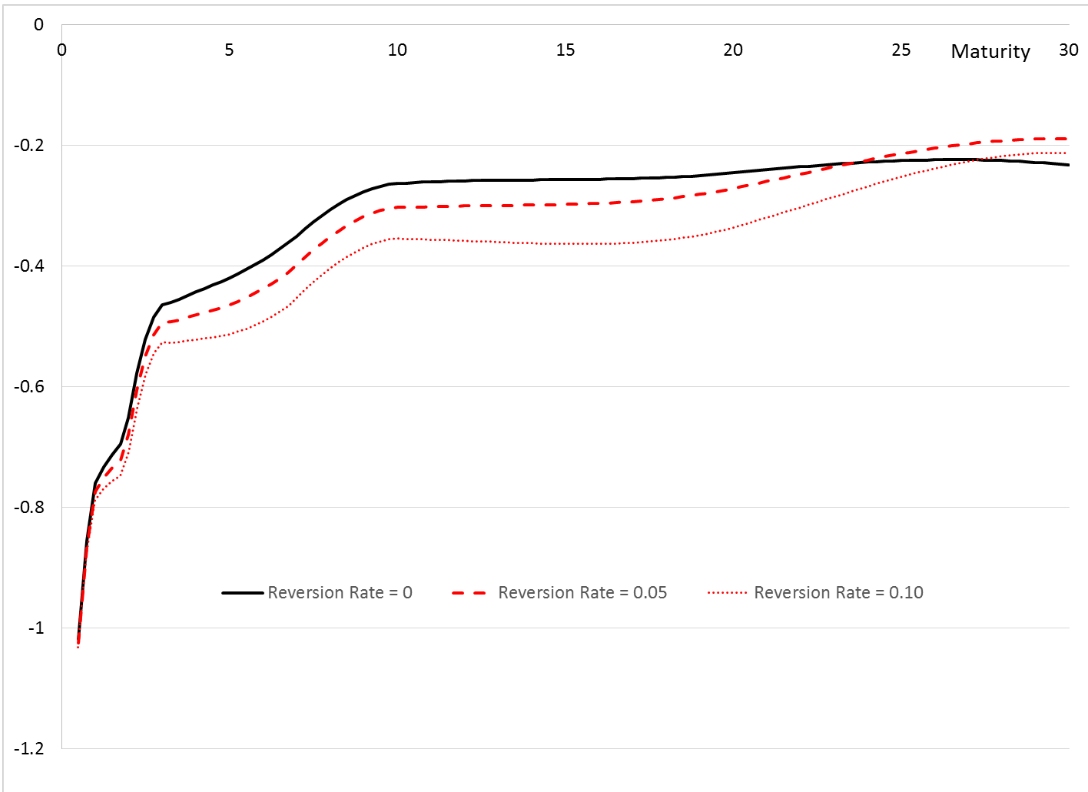

Figure 3

Distribution of the Short-term Interest Rate in 1 Year
Estimated from Data on December 2, 2013

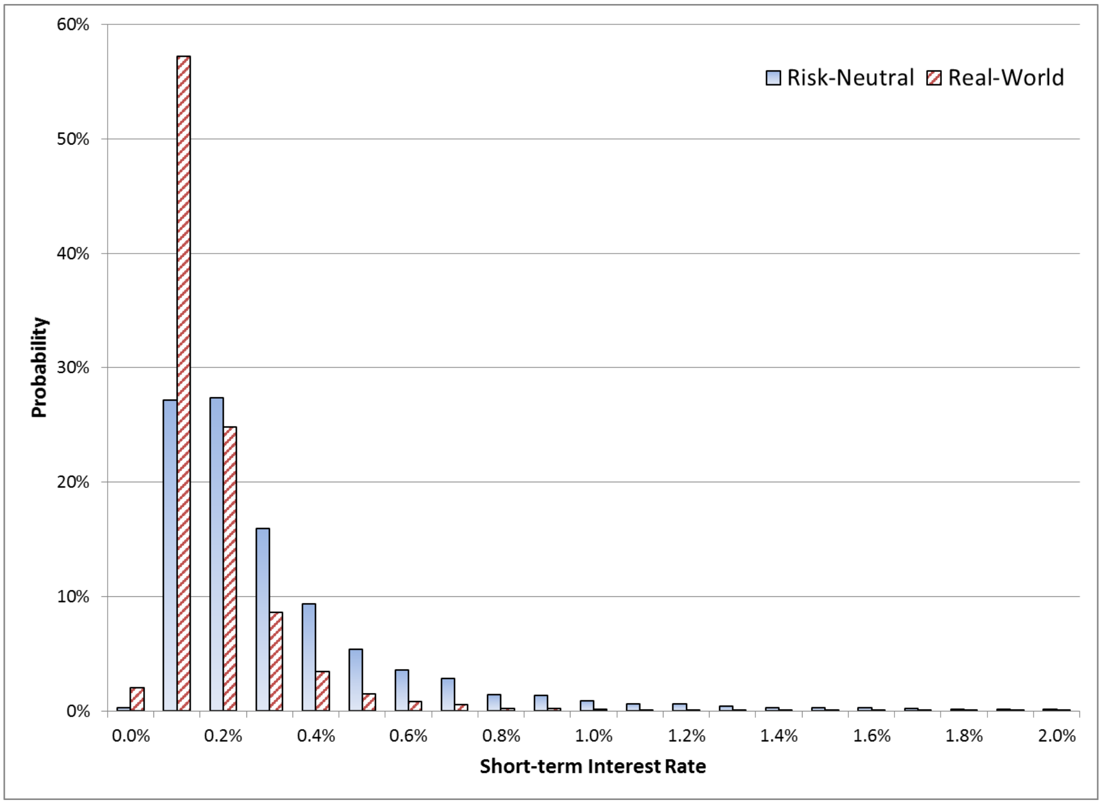

Figure 4

Distribution of the Short-term Interest Rate in 5 Years
Estimated from Data on December 2, 2013

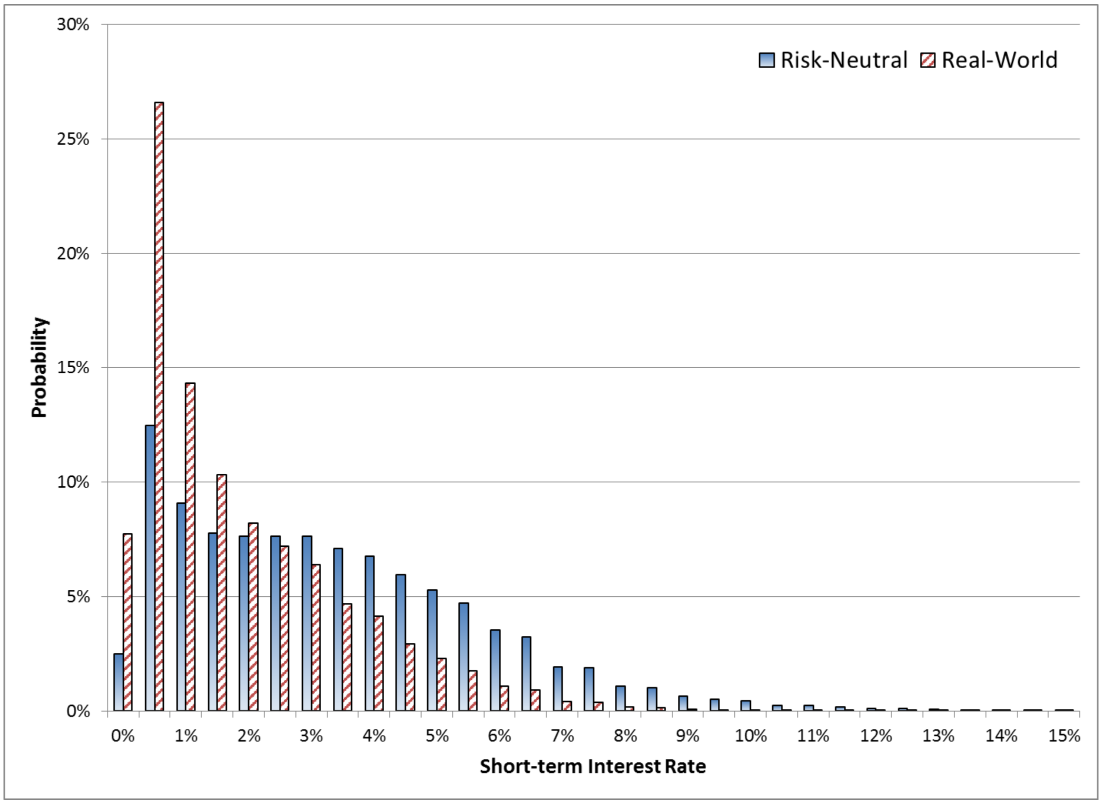

Figure 5

Distribution of the Short-term Interest Rate in 30 Years
Estimated from Data on December 2, 2013

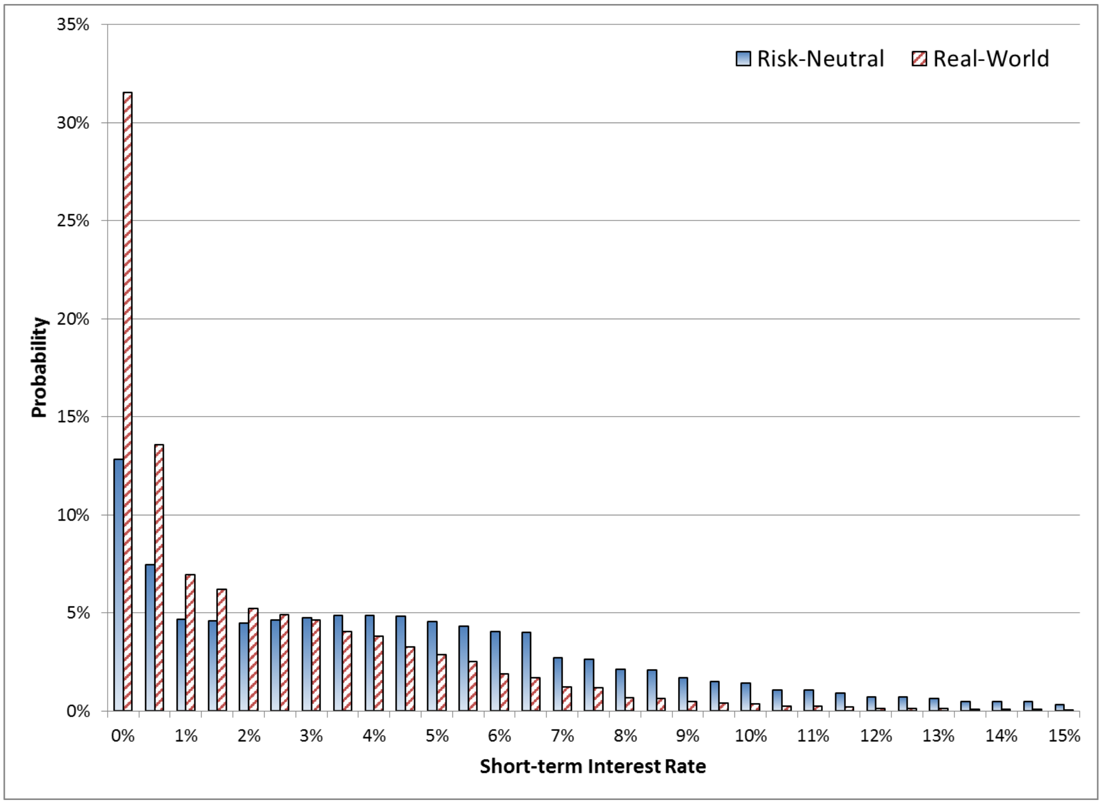

Figure 6

Expected Future Short-term Interest Rate
Estimated from Data on December 2, 2013

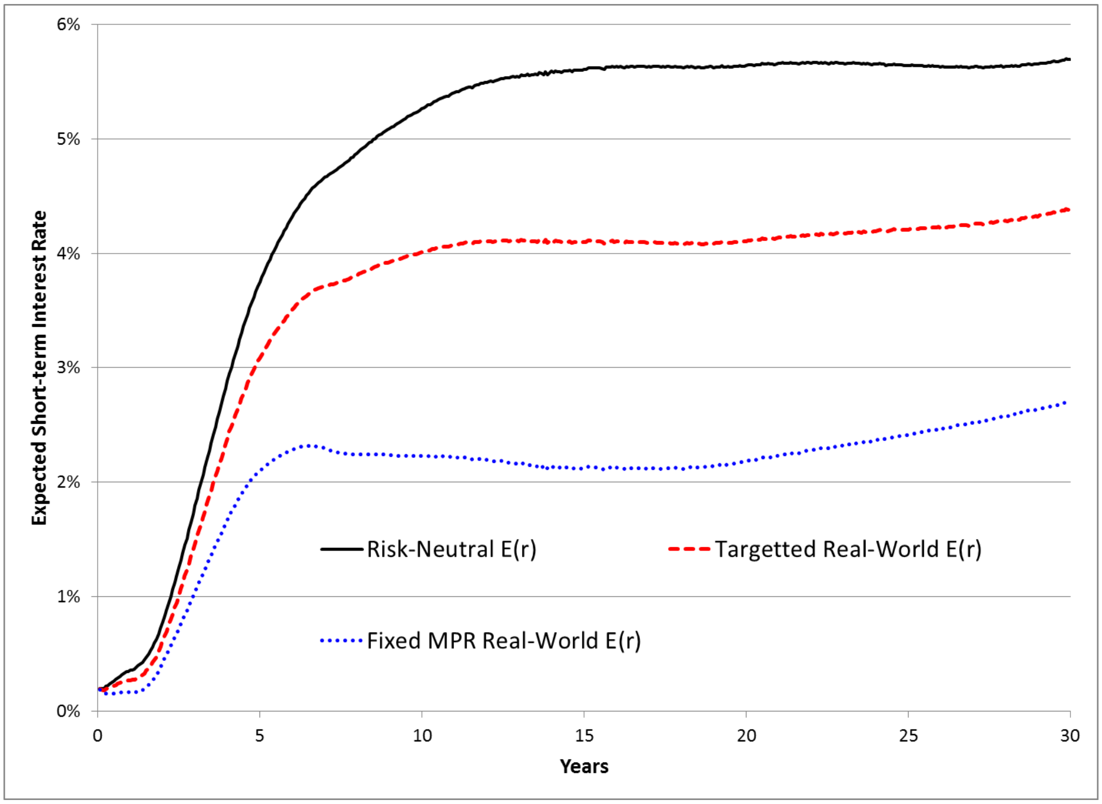

Figure 7

Standard Deviation of Future Short-term Interest Rate
Estimated from Data on December 2, 2013