A METHODOLOGY FOR ASSESSING MODEL RISK AND ITS APPLICATION TO THE IMPLIED VOLATILITY FUNCTION MODEL

Forthcoming: *Journal of Financial and Quantitative Analysis*

John Hull and Wulin Suo*

This Version: July, 2001

* Joseph L. Rotman School of Management, University of Toronto, 105 St George Street, Toronto, Ontario, Canada M5S 3E6, and School of Business, Queen's University, 99 University Avenue, Kingston, Ontario, Canada K7L 3N6. We are grateful for helpful comments from Phelim Boyle, Angelo Melino, Alan White, two anonymous referees, and seminar participants at the Courant Institute, Hong Kong University, University of Toronto, and Queen's University

A METHODOLOGY FOR ASSESSING MODEL RISK AND ITS APPLICATION TO THE IMPLIED VOLATILITY FUNCTION MODEL

Abstract

We propose a methodology for assessing model risk and apply it to the implied volatility function (IVF) model. This is a popular model among traders for valuing exotic options. Our research is different from other tests of the IVF model in that we reflect the traders' practice of using the model for the relative pricing of exotic and plain vanilla options at one point in time. We find little evidence of model risk when the IVF model is used to price and hedge compound options. However, there is significant model risk when it is used to price and hedge some barrier options.

I. Introduction

In the 1980s and 1990s we have seen a remarkable growth in both the volume and the variety of the contracts that are traded in the over-the-counter derivatives market. Banks and other financial institutions rely on mathematical models for pricing and marking to market these contracts. As a result they have become increasingly exposed to what is termed "model risk". This is the risk arising from the use of an inadequate model. Most large financial institutions currently set aside reserves for model risk. This means that they defer recognition of part of their trading profits when the profits are considered to be dependent on the pricing model used.

Bank regulators currently require banks to hold capital for market risk and credit risk. The Basel Committee on Banking Supervision has indicated that it intends to expand the scope of the current capital adequacy framework so that it more fully captures the risks to which a bank is exposed.[1] In particular, it plans to require capital for operational risk, one important component of which is model risk. As the Basel Committee moves towards implementing its proposed new framework, banks are likely to come under increasing pressure to identify and quantify model risk.

As pointed out by Green and Figlewski (1999), an inadequate model may lead to a number of problems for a financial institution. It may cause contracts to be sold for too low a price or purchased for too high a price. It may also lead to a bad hedging strategy being used or cause market risk and credit risk measures to be severely in error.

Derman (1996) lists a number of reasons why a model may be inadequate. It may be badly specified; it may be incorrectly implemented; the model parameters may not be estimated correctly; and so on. In this paper we focus on model risk arising from the specification of the model.

The specification of the model is not usually a significant issue in the pricing and

[1] See Basel Committee for Banking Supervision (1999).

hedging of "plain vanilla" instruments such as European options on stock indices and currencies, and U.S. dollar interest rate caps and floors. A great deal of information on the way these instruments are priced by the market at any given time is available from brokers and other sources. As a result, the pricing and hedging of a new or an existing plain vanilla instrument is largely model independent.

Consider, for example, European options on the S&P 500. Most banks use a relative value approach to pricing. Each day data is collected from brokers, exchanges, and other sources on the Black–Scholes implied volatilities of a range of actively traded plain vanilla options. Interpolation techniques are then used on the data to determine a complete function (known as the implied volatility surface) relating the Black–Scholes implied volatility for a European option to its strike price and time to maturity. This volatility surface enables the price of any European option to be calculated. When hedging traders attempt to protect themselves against possible changes in the implied volatility surface as well as against changes in the index. As a result model misspecification has much less impact on the hedging effectiveness than would be the case if the traders relied solely on delta hedging.

Consider what would happen if a trader switched from Black–Scholes to another model (say, the constant elasticity of variance model) to price options on the S&P 500. Volatility parameters and the volatility surface would change. But, if the model is used for relative pricing in the same way as Black–Scholes, prices would change very little and hedging would be similar. The same is true for most other models that are used by traders in liquid markets.

Model risk is a much more serious issue when exotic options and structured deals are priced. Unfortunately the measurement of model risk for these types of instruments is not easy. The traditional approach of comparing model prices with market prices cannot be used because market prices for exotic options are not readily available. (Indeed, if market prices were readily available, there would be very little model risk.) One approach used by researchers is to test directly the stochastic processes assumed by the model for market variables. This often involves estimating the parameters of the stochastic process from sample data and then testing the process out of sample. The goal of the approach is the development of improved stochastic processes that fit historical data well and are evolutionary realistic. Unfortunately however, the approach is of limited value as a test of model risk. This is because, in assessing model risk we are interested in estimating potential errors in a particular model as it is used in the trading room. Traders calibrate their models daily, or even more frequently, to market data. Any test that assumes the model is calibrated once to data and then used unchanged thereafter is liable to estimate model risk incorrectly.

Some researchers have tested the effectiveness of a model when it is calibrated to market data at one point in time, t_1, and used to price options at a later time, t_2. Dumas, Fleming, and Whaley (1998) use this approach to test the implied volatility function model for equity options. Recently a number of other papers have used the approach in interest rate markets. For example, Gupta and Subrahmanyam (2000) look at interest rate caps with multiple strike prices and find that, while one-factor models are adequate for pricing, two-factor models are necessary for hedging. Driessen, Klaassen, and Melenberg (2000) look at both caps and swap options and find that the difference between the use of one-factor and two-factor models for hedging disappears when the set of hedge instruments covers all maturities at which there is a payout.

These papers are interesting but still do not capture the essence of the risk in models as they are used in trading rooms. As mentioned earlier, model risk is a concern when exotic options are priced. Traders typically use a model to price a particular exotic option in terms of the market prices of a number of plain vanilla options at a particular time. They calibrate the model to plain vanilla options at time t_1 and use the model to price an exotic option at the same time, t_1. Whether the model does well or badly pricing other plain vanilla options at a later time, t_2, is not necessarily an indication of its ability to provide good relative pricing for exotic and plain vanilla options at time t_1. [2]

Given the difficulties we have mentioned in using traditional approaches as tests of model risk, we propose the following approach for testing the applicability of a model for valuing an exotic option:

1. Assume that prices in the market are governed by a plausible multi-factor no-arbitrage model (the "true model") that is quite different from, and more complex than, the model being tested.
2. Determine the parameters of the true model by fitting it to representative market data.
3. Compare the pricing and hedging performance of the model being tested with the pricing and hedging performance of the true model for the exotic option. When the market prices of plain vanilla options are required to calibrate the model being tested, generate them from the true model.

If the pricing and hedging performance of the trader's model is good when compared to the assumed true model, we can be optimistic (but not certain) that there is little model risk. If the pricing and hedging performance is poor, there is evidence of model risk and grounds for setting aside reserves or using a different model.

In some ways our approach is in the spirit of Green and Figlewski (1999). These authors investigate how well traders perform when they use a Black–Scholes model that is recalibrated daily to historical data. However, our research is different in that we assume the model is calibrated to current market data rather than historical data. Also we use the model for pricing exotic options rather than vanilla options.

Research that follows an approach most similar to our own is Andersen and Andreasen (2001) and Longstaff, Santa-Clara, and Schwartz (2001). These authors test the effective-

[2] Hull and White (1999) get closer to the way models are used in practice by investigating the relative pricing of caps and swaptions at a particular point in time. However, they do not look at exotic options.

ness of a one-factor interest-rate model rate in pricing Bermudan swap options in a world where the term structure is driven by more than one factor. The one-factor model is recalibrated daily to caps or European swap options or both.

We illustrate our approach by using it to assess model risk in the implied volatility function model, which is popular among traders for valuing exotic options. Section II describes the model and the way it is used by traders. Section III explains potential errors in the model. Section IV describes our pricing tests for compound options and barrier options. Section V describes our hedging tests. Section VI discusses the economic intuition for our results. Conclusions are in Section VII.

II. The Implied Volatility Function Model

Many models have been developed to price exotic options on equities and foreign currencies using the Black–Scholes assumptions. Examples are Geske's (1979) model for pricing compound options, Merton's (1973) model for pricing barrier options, and Goldman et al's (1979) model for lookback options. If the implied volatilities of plain vanilla options were independent of strike price so that the Black–Scholes assumptions applied, it would be easy to use these models. In practice, the implied volatilities of equity and currency options are found to be systematically dependent on strike price. Authors such as Rubinstein (1994) and Jackwerth and Rubinstein (1996) show that the implied volatilities of stock and stock index options exhibit a pronounced "skew". For options with a particular maturity, the implied volatility decreases as the strike price increases. For currency options, this skew becomes a "smile". For a given maturity, the implied volatility of an option on a foreign currency is a U-shaped function of the strike price. The implied volatility is lowest for an option that is at or close to the money. It becomes progressively higher as the option moves either in or out of the money.

It is difficult to use models based on the Black–Scholes assumptions for exotic options because there is no easy way of determining the appropriate volatility from the volatility surface for plain vanilla options. More elaborate models than those based on the Black–Scholes assumptions have been developed. For example, Merton (1976) and Bates (1996) have proposed jump-diffusion models. Heston (1993), Hull and White (1987, 1988), and Stein and Stein (1995) have proposed models where volatility follows a stochastic process. These models are potentially useful for pricing exotic options because, when parameters are chosen appropriately, the Black–Scholes implied volatilities calculated from the models have a similar pattern to those observed in the market. However, the models are not widely used by traders. Most traders like to use a model for pricing exotic options that exactly matches the volatility surface calculated from plain vanilla options. Research by Rubinstein (1994), Derman and Kani (1994), Dupire (1994), and Andersen and Brotherton–Ratcliffe

(1998) shows how a one-factor model with this property can be constructed by making volatility a deterministic function of the asset price and time. We will refer to this model as the implied volatility function (IVF) model.[3]

The risk-neutral process followed by the asset price, S, in the IVF model is

$$\frac{dS}{S} = [r(t) - q(t)]dt + \sigma(S,t)dz \tag{1}$$

where $\sigma(S,t)$ is the volatility of S. The variable $r(t)$ is the risk-free rate at time t and $q(t)$ is the dividend yield at time t. (When a currency is modeled, it can be treated as an asset providing a dividend yield equal to the foreign risk-free rate.) Derivatives dependent on the asset price satisfy the differential equation

$$\frac{\partial f}{\partial t} + [r(t) - q(t)]\frac{\partial f}{\partial S} + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 f}{\partial S^2} = r(t)f \tag{2}$$

The variables $r(t)$ and $q(t)$ in equation (2) are set equal to the instantaneous forward risk-free rate and instantaneous forward dividend yield for a maturity t respectively. As shown by Dupire (1994) and Andersen and Brotherton–Ratcliffe (1998), the volatility function, $\sigma(S,t)$ can be determined analytically from the prices of European options with different strike prices and times to maturity. If we write $c_{\text{mkt}}(K,T)$ as the market price of a call option with strike price K and maturity T, then:

$$\sigma(K,T)^2 = 2\frac{\partial c_{\text{mkt}}/\partial T + q(T)c_{\text{mkt}} + [r(T) - q(T)]K\partial c_{\text{mkt}}/\partial K}{K^2 \partial^2 c_{\text{mkt}}/\partial K^2} \tag{3}$$

Dumas, Fleming and Whaley (1998) provide the most complete tests of the IVF model to date. They show that there are significant errors when the IVF model is fitted to the market at a particular time and then used to price options one week later. They also find

[3] Recently Brace et al (2001), Ledoit and Santa Clara (1998), and Suo (2001) have proposed an extension to the model where the volatility function is a stochastic function of the asset price and time. They derive no-arbitrage conditions for the process followed by the volatility function.

that the difference between the observed and predicted option prices is larger for complex parameterizations of the volatility functions than for a constant volatility specification.

These tests are interesting in that they provide evidence of the need for traders to change the parameters of the IVF model on a regular basis in order to match market prices. They also show that the IVF model does not capture the dynamics of European option prices. However, as pointed out in Section I, they do not determine whether the IVF model provides good relative pricing for an exotic option and a set of plain vanilla option at one particular point in time. Our research investigates this question.

A. The Trader's IVF Model

At this stage it is appropriate to make some observations about the nature of the IVF model (and most other models) as they are actually used in the trading room.

In practice, pricing an exotic option at a particular time using the IVF model involves two steps:

Step 1: Calibrate the model so that $r(t)$, $q(t)$, and $\sigma(S,t)$ are consistent with current data on the volatility surface, interest rates, and dividends yields.

Step 2: Use equations (2) and (3) to value the exotic option

It is interesting to ask what model are traders really using when they do this. It is not the model in equation (1) because, as is evident from the research of Dumas, Fleming and Whaley (1998), estimates of the function $\sigma(S,t)$ change frequently.

If we assume for simplicity that interest rates and dividends are deterministic, the trader's version of the IVF model has the form

$$P = g(S, t, c_1, c_2, \ldots, c_n, \theta_1, \theta_2, \ldots, \theta_m) \tag{4}$$

where P is the price of an exotic option and g is a function. The variables c_i $(1 \le i \le n)$ are the prices of the European options used to define the volatility surface when the model is being calibrated. The variables θ_j $(1 \le j \le m)$ are constant parameters defining the exotic option.

The model in equation (4) has $n+1$ stochastic variables.[4] It has two desirable properties. First, it satisfies the boundary conditions for P. Second, it prices all European options (and all linear combinations of European options) exactly consistently with the market at all times.[5] The model in equation (1) is used as a tool for creating a model with these properties.

Define a model that is recalibrated to market data daily or even more frequently as a "Continual Recalibration" (CR) models and a model that is calibrated just once as "Single Calibration" (SC) models. We will refer to the IVF model in equation (1) as the SC-IVF model and the CR version of the IVF model as the CR-IVF model. One of the features of derivatives markets is that most models are developed by researchers as SC models and then used by traders as CR models. It is interesting to note that although an SC model is almost invariably developed as a no-arbitrage model, there is no guarantee that the corresponding CR model will be a no-arbitrage model.

[4] All $n+1$ securities may not in practice be necessary to span the pricing kernel, but recent research by Buraschi and Jackwerth (2001) clearly indicates that one security is not enough for this purpose.

[5] Investment banks consider this property to be of great practical importance. If a model does not price plain vanilla instruments consistently with the market at all times, a bank's own traders will find a way to arbitrage the bank's model to generate short term profit for the book they are managing.

III. Potential Errors in the CR-IVF Model

The CR-IVF model is designed so that it always values all European options correctly at all times. This means that the risk-neutral probability distribution of the asset price at any future time is always correct. This in turn means that the CR-IVF model always correctly prices a derivative when its payoff is contingent on the asset price at only one time. For example, it correctly prices cash-or-nothing calls and puts and asset-or-nothing calls and puts. However, there is no guarantee that the CR-IVF model correctly prices derivatives where the payoff is contingent on the asset price at more than one time. This is because the joint distribution of the asset price at times T_1 and T_2 is not uniquely determined from the marginal distributions at times T_1 and T_2.

To express this more formally, define $\phi_n[t_1, t_2, \ldots, t_n]$ as the joint probability distribution of the asset price at times t_1, t_2, ..., t_n. The CR-IVF model is designed so that $\phi_1(t_1)$ is correct for all t_1, but this does not ensure that $\phi_n[t_1, t_2, \ldots, t_n]$ is correct for $n > 1$. This point is also made by Britten–Jones and Neuberger (2000) who produce some interesting results characterizing the set of all continuous price processes that are consistent with a given set of option prices.

Consider a call-on-call compound option. This is an option where the holder has the right at time T_1 to pay K_1 and obtain a European call option to purchase the asset for a price K_2 at time T_2 ($T_2 > T_1$). If we denote by C_1 the price at time T_1 of a European call option with maturity T_2 and strike K_2 then, assuming a constant interest rate r, the price of the call-on-call option can be written as

$$\begin{aligned}
&e^{-rT_1} E_0 \left[(C_1 - K_1)^+ \right] \\
&= e^{-rT_1} E_0 \left[(e^{-r(T_2 - T_1)} E_1((S_2 - K_2)^+) - K_1) \mid C_1 \geq K_1 \right] \\
&= e^{-rT_2} E_0 \left[S_2 \mid S_2 \geq K_2, C_1 \geq K_1 \right] \\
&\quad - K_2 e^{-rT_2} P(S_2 \geq K_2, C_1 \geq K_1) - K_1 e^{-rT_1} P(C_1 \geq K_1).
\end{aligned}$$

where E_0 and E_1 denote risk-neutral expectations at times zero and T_1 respectively and P is the risk-neutral probability measure. This shows that the option price is dependent on $\phi_2(T_1,\, T_2)$.[6] For this reason, there is potential for model risk.

In the case of other derivatives, the dependence of the payoff on the joint probability distribution of the underlying asset price at different times is much more complex. Consider an up-and-out barrier call option with strike price K maturing at time T_N where the asset price is observed at times $T_1,\ T_2,\ \ldots,\ T_N$ for the purposes of determining whether the barrier has been hit. In this case, when interest rates are constant, the price of the option is

$$
\begin{aligned}
& e^{-rT_N} E_0\left[(S_N - K)^+ \mid S_1 < T_1, S_2 < T_2, \ldots, S_N < T_N\right] \\
&= e^{-rT_N} E_0\left[S_N \mid S_1 < T_1, S_2 < T_2, \ldots, S_{N-1} < T_{N-1}, K \le S_N < T_N\right] \\
&\quad - K e^{-rT_N} P(S_1 < T_1, S_2 < T_2, \ldots, S_{N-1} < T_{N-1}, K \le S_N < T_N)
\end{aligned}
$$

where S_i is the asset price at time T_i. This shows that the price of the option depends in a complex way on $\phi_N[T_1, T_2, \ldots, T_N]$. As mentioned above, the CR-IVF model is designed so that $\phi_1(T_i)$ is correct for $1 \le i \le N$, but this does not unambiguously determine $\phi_N[T_1, T_2, \ldots, T_N]$. Again there is potential for model risk.

[6] To be more precise, it is dependent on the joint distribution of the asset price at time T_2 and relevant state variables at time T_1. An example of a potentially relevant state variable at time T_1, other that the asset price, is the asset price volatility.

IV. Pricing Tests

In this section we test the performance of the CR-IVF model when it is used to price exotic options on stock indices and exchange rates. These assets are natural choices for our tests because in practice the IVF model is frequently used to value exotic options on them. As mentioned in Section II, the volatility surface for options on an exchange rate is quite different from that for options on an equity index. This means that the model's performance may be different for the two types of assets.

As explained in Section I our test of model risk involves assuming that market prices are determined by a no-arbitrage model that is more complex than, and quite different from, the model under consideration. It is desirable to choose a no-arbitrage model that reflects empirical research. Bakshi, Cao, and Chen (1997) who look at options on the S&P 500 conclude "Our empirical evidence indicates that regardless of the performance yardstick, taking stochastic volatility into account is of first-order importance in improving on the Black–Scholes formula." Researchers such as Melino and Turnbull (1995) looking at foreign exchange options have reached similar conclusions. Buraschi and Jackwerth's (2001) research also indicates that stochastic volatility is an important factor in option pricing. We therefore choose to test the CR-IVF model using a two-factor stochastic volatility model.[7]

The two-factor stochastic volatility model we assume is:

$$\frac{dS}{S} = (r - q)dt + v dz_S \tag{5}$$

$$dv = \kappa(\theta - v)dt + \xi dz_v \tag{6}$$

[7] There would be no point in choosing a one-factor stochastic volatility model such as the constant elasticity of variance model because we know from the structure of equation (1) that the IVF model can exactly match such a model. Also note that if the true model were a stochastic volatility model with time varying parameters, our approach does provide a test of the ability of the CR-IVF model to provide good relative pricing as we are only concerned with the parameters at one point in time.

In these equations z_S and z_v are Wiener processes with an instantaneous correlation ρ. The variable v is a factor driving asset prices. The volatility of S is $|v|$ and the model is well defined when v is negative. The parameters κ, θ, and ξ are the mean-reversion rate, long-run average volatility, and standard deviation of v, respectively, and are assumed to be constants. We also assume the spot rate, r, and the yield on the asset, q, are constants. The parameters of the model are chosen to minimize the root mean square error in matching the observed market prices for European options. The model is similar to the one proposed by Heston (1993) and has similar analytic properties.[8]

A valuation formula for the European call option price, $c_{\mathrm{sv}}(S, v, t; K, T)$, in the model can be computed through the inversion of characteristic functions of random variables. It takes the form:

$$c_{\mathrm{sv}}(S, v, t; K, T) = e^{-q(T-t)} S(t) F_1 - e^{-r(T-t)} K F_2 \tag{7}$$

where F_1 and F_2 are integrals that can be evaluated efficiently using numerical procedures such as quadrature. More details on the model can be found in Schöbel and Zhu (1998).[9]

When considering S&P 500 options we chose model parameters to provide as close a fit as possible to the volatility surface for the S&P 500 reported in Andersen and Brotherton–Ratcliffe (1998). The parameters are

---

[8] If $V = v^2$ is the variance rate, the model we assume implies that

$$dV = (\xi + 2\kappa\theta\sqrt{V} - 2\kappa V)\,dt + 2\xi\sqrt{V}\,dz_v$$

Heston's model is of the form:

$$dV = (\alpha - \beta V)\,dt + \gamma\sqrt{V}\,dz_v$$

More details on the model's analytic properties are available from the authors on request.

[9] The results in Schöbel and Zhu (1998) and Monte Carlo simulation were used as a check of the implementation of equation (7).

$r = 5.9\%$, $q = 1.4\%$, $v(0) = 0.25$, $\kappa = 0.16$, $\theta = 0.3$, $\xi = 0.09$, and $\rho = -0.79$.

The correlation between the equity index and its volatility is highly negative. This arises from the steep volatility skew in the Andersen and Brotherton–Ratcliffe data. As Hull and White (1987, 1988) show, a negative correlation leads to a volatility skew with the skew becoming steeper as the correlation becomes more negative.[10]

When considering foreign exchange options we chose parameters to provide as close as fit as possible to the volatility surface for the U.S. dollar-Swiss franc exchange rate provided to us by a large U.S. investment bank. The parameters are

$r = 5.9\%$, $q = 3.5\%$, $v(0) = 0.13$, $\kappa = 0.11$, $\theta = 0.1$, $\xi = 0.04$, and $\rho = 0.16$.

Note that the magnitude of the correlation is much less in this case. As explained in Hull and White (1987, 1988) a low correlation gives rise to a volatility smile similar to that observed in the foreign exchange markets. The volatility surfaces given by the two parameter sets are shown in Tables 1 and 2.

Our pricing tests of the CR-IVF model consist of the following steps:

Step 1: Price an exotic option using the best-fit stochastic volatility model. We denote this price by f_{sv}.

Step 2: Fit the IVF model to the market prices of European call options that are given by the best-fit stochastic volatility model.

Step 3: Use the IVF model to price the same exotic option. We denote this price by f_{ivf}.

Step 4: Use the Black–Scholes assumptions to price the exotic option. We denote this price by f_{bs}.

Step 5: Compare f_{sv}, f_{ivf}, and f_{bs}.

We calculate the market prices of European call options, c_{mkt}, using equation (7). We fit

[10] We fitted the model to 30 different volatility surfaces provided to us by a major broker for month ends between June 1998 and November 2000. The parameter values we use here are typical of those obtained from this data. For example, the best fit value of ρ was always between –0.6 and –0.9.

the IVF model to these prices by calculating $\partial c_{\text{mkt}}/\partial t$, $\partial c_{\text{mkt}}/\partial K$, and $\partial^2 c_{\text{mkt}}/\partial K^2$ from equation (7) and then using equation (3).

We consider two types of exotic options: a call-on-call compound option and a knock-out barrier option. We use Monte Carlo simulation with 300 time steps and 100,000 trials to estimate the prices of these options for the stochastic volatility model.[11] For this purpose, equations (5) and (6) are discretized to

$$\ln \frac{S_{i+1}}{S_i} = \left(r - q - \frac{v_i^2}{2}\right)\Delta t + v_i \epsilon_1 \sqrt{\Delta t} \tag{8}$$

$$v_{i+1} - v_i = \kappa(\theta - v_i)\Delta t + \xi \epsilon_2 \sqrt{\Delta t} \tag{9}$$

where Δt is the length of the Monte Carlo simulation time step, S_i and v_i are the asset price and its volatility at time $i\Delta t$, and ϵ_1 and ϵ_2 are random samples from two unit normal distributions with correlation, ρ.

We estimate the prices given by the IVF model from equation (2) using the implicit Crank-Nicholson finite difference method described in Andersen and Brotherton–Ratcliffe (1998). This involves constructing a 120×70 rectangular grid of points in (x, t)-space, where $x = \ln S$. The grid extends from time zero to the maturity of the exotic option, T_{mat}. Define x_{min} and x_{max} as the lowest and highest x-values considered on the grid. (We explain how these are determined later.) Boundary conditions determine the values of the exotic option on the $x = x_{\text{max}}$, $x = x_{\text{min}}$ and $t = T_{\text{mat}}$ edges of the grid. The differential equation (2) enables relationships to be established between the values of the exotic option at the nodes at the ith time point and its values at the nodes at the $(i + 1)$th time point. These relationships are used in conjunction with boundary conditions to determine the value of the exotic option at all interior nodes of the grid and its value at the nodes at time zero.

[11] To reduce the variance of the estimates, we use the antithetic variable technique described in Boyle (1977).

The exotic options we chose for testing the CR-IVF model are compound options and barrier options. As explained in Section III compound options are dependent on the joint distribution of the asset price at just two points in time. They are therefore an ideal test of the ability of the IVF model to recover a relatively simple joint distribution. If the IVF model does not work well for compound options it is unlikely to be appropriate for the vast majority of exotic options that are traded. Barrier options were chosen because they are very popular exotic options and, as explained in Section III are dependent in a complex way on the joint distribution of asset prices at many different points in time. The discussion in Section III leads us to conjecture that the IVF model will work less well for barrier options than compound options.

A. Compound Options

A call-on-call compound option is an option where the holder has the right at time T_1 to pay K_1 and obtain a European call option to purchase the asset for a price K_2 at time T_2 ($T_2 > T_1$). When using Monte Carlo simulation to calculate f_{sv}, each trial involves using equations (8) and (9) to calculate the asset price and its volatility at time T_1. It is not necessary to simulate beyond time T_1 because the value of a European call option with strike price K_2 and maturity T_2 can be calculated at time T_1 using equation (7). Define $S_{1,j}$ and $v_{1,j}$ as the asset price and volatility at time T_1 on the jth trial, and $w_{1,j}$ as the value at time T_1 of a call option with strike price K_2 maturing at T_2 for the jth trial. It follows that

$$w_{1,j} = c_{\text{sv}}(S_{1,j}, v_{1,j}, T_1, K_2, T_2)$$

The estimate of the value of the option given by the stochastic volatility model is:

$$f_{\text{sv}} = \frac{e^{-rT_1}}{N} \sum_{j=1}^{N} \max(w_{1,j} - K_1,\, 0)$$

We calculate the IVF price for the compound option by building the finite difference grid out to time T_2. Between times T_1 and T_2, we use the grid to calculate the price, w, of a European call option with strike price K_2 maturing at time T_2. This enables the value of the compound option at the nodes at time T_1 to be calculated as $\max(w - K_1, 0)$. We then use the part of the grid between time zero and time T_1 to calculate the value of the compound option at time zero. We set $x_{\min} = \ln S_{\min}$ and $x_{\max} = \ln S_{\max}$ where $S_{\min}$ and $S_{\max}$ are very high and very low asset prices, respectively. The boundary conditions we use are:

$$w = \max(e^x - K_2, 0) \text{ when } t = T_2$$

$$w = 0 \text{ when } x = x_{\min} \text{ and } T_1 \le t \le T_2$$

$$w = e^x - K_2 e^{-r(T_2-t)} \text{ when } x = x_{\max} \text{ and } T_1 \le t \le T_2$$

$$f_{\text{ivf}} = 0 \text{ when } x = x_{\min} \text{ and } 0 \le t \le T_1$$

$$f_{\text{ivf}} = e^x - K_2 e^{-r(T_2-t)} - K_1 e^{-r(T_1-t)} \text{ when } x = x_{\min} \text{ and } 0 \le t \le T_1$$

The value of a compound option using the Black–Scholes assumptions was first produced by Geske (1979). Geske shows that at time zero:

$$f_{\text{bs}} = S(0)e^{-qT_2} M(a_1, b_1; \sqrt{T_1/T_2}) - K_2 e^{-rT_2} M(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(a_2)$$

where

$$a_1 = \frac{\ln[S(0)/S^*] + (r - q + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1}$$
$$b_1 = \frac{\ln[S(0)/K_2] + (r - q + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2}$$

and $M(a, b; \rho)$, is the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b when the coefficient of correlation between the variables is ρ. The variable S^* is the asset price at time T_1 for which the price at time T_1 of a European call option with strike price K_2 and maturity T_2 equals K_1. If the actual asset price is above S^* at time T_1, the first option will be exercised; if it is not above S^*, the compound option expires worthless. In computing f_{bs} we set σ equal to the implied volatility of a European option maturing at time T_2 with a strike price of K_2.

Table 3 shows f_{sv} and the percentage errors when the option price is approximated by f_{ivf} and f_{bs} for the case where $T_1 = 1$, $T_2 = 2$, and K_2 equals the initial asset price. It considers a wide range of values of K_1. The table shows that the IVF model works very well. For compound options where the stochastic volatility price is greater than 1% of the initial asset price, the IVF price is within 2% of the stochastic volatility price. When very high strike prices are used with Parameter Set II this percentage error is higher, but this is because the stochastic volatility price of the compound option is very low. Measured as a percent of the initial asset price the absolute pricing error of the IVF model is never greater than 0.08%.

The Black–Scholes model, on the other hand, performs quite badly. For high values of the strike price, K_1, it significantly overprices the compound option in the case of the stock index data and significantly underprices it in the case of the foreign currency data. The reason is that, when K_1 is high, the first call option is exercised only when the asset price is very high at time T_1. Consider first the stock index data. As shown in Table 1, the implied volatility is a declining function of the strike price. (This is the volatility skew phenomenon for a stock index described in Section I). As a result the probability distribution of the asset price at time T_1 has a more heavy left tail and a less heavy right tail than a lognormal distribution when the latter is calculated using the at-the-money volatility, and very high asset prices are much less likely than they are under the Black–Scholes model. This means that the first option is much more likely to be exercised at time T_1 in the Black–Scholes world than in the assumed true world. Consider next the foreign currency data. As shown in Table 2, the implied volatility is a U-shaped function. (This is the volatility smile phenomenon for a currency described in Section I.) The results in the probability distribution of the asset price having heavier left and right tails than a lognormal distribution when the latter is calculated using the at-the-money volatility, and very high asset prices are much more likely than they are under the Black–Scholes model. This means that the first option is much less likely to be exercised in the Black–Scholes world than in the assumed true world.

Traders sometimes try to make the Black–Scholes model work for compound options by adjusting the volatility. Sometimes they use two different volatilities, one for the period between time zero and time T_1 and the other for the period between time T_1 and time T_2. There is of course some volatility (or pair of volatilities) that will give the correct price for any given compound option. But the price of a compound option given by the Black-Scholes model is highly sensitive to the volatility and any procedure that involves estimating the "correct" volatility is dangerous and liable to give rise to significant errors.

Based on the tests reported here and other similar tests we have carried out, the IVF model is a big improvement over the Black–Scholes model when compound options are priced. There is very little evidence of model risk. This is encouraging, but of course it provides no guarantee that the model is also a proxy for all more complicated multifactor models.

B. Barrier Options

The second exotic option we consider is a knock-out barrier call option. This is a European call option with strike price K and maturity T that ceases to exist if the asset price reaches a barrier level, H. When the barrier is greater than the initial asset price, the option is referred to as an up-and-out call; when the barrier is less than the initial asset price, it is referred to as a down-and-out call.

When using Monte Carlo simulation to calculate f_{sv}, each trial involved using equations (8) and (9) simulate a path for the asset price between time zero and time T. For an up-and-out (down-and-out) option, if for some i, the asset price is above (below) H at time $i\Delta t$ on the jth trial the payoff from the barrier option is set equal to zero on that trial. Otherwise the payoff from the barrier option is $\max[S(T) - K, 0]$ at time T. The estimate of f_{sv} is the arithmetic mean of the payoffs on all trials discounted from time T to time zero at rate r.[12]

[12] To improve computational efficiency we applied the correction for discrete observations

We calculate the IVF price for the barrier option by building the finite difference grid out to time T. In the case of an up-and-out option, we set $x_{\max} = \ln(H)$ and $x_{\min} = \ln S_{\min}$ where $S_{\min}$ is a very low asset price; in the case of a down-and-out option, we set $x_{\min} = \ln(H)$ and $x_{\max} = \ln S_{\max}$ where $S_{\max}$ is a very high asset price. For an up-and-out call option, the boundary conditions are:

$$f_{\text{ivf}} = \max(e^x - K_2, 0) \text{ when } t = T$$

$$f_{\text{ivf}} = 0 \text{ when } x \geq \ln(H) \text{ and } 0 \leq t \leq T$$

$$f_{\text{ivf}} = 0 \text{ when } x = x_{\min} \text{ and } 0 \leq t \leq T$$

For a down-and-out call, the boundary conditions are similar except that

$$f_{\text{ivf}} = e^x - K_2 e^{-r(T-t)} \text{ when } x = x_{\max}$$

The value of knock-out options using the Black–Scholes assumptions was first produced by Merton (1973). He showed that at time zero, the price of a down-and-out call option is

$$f_{\text{bs}} = S(0)N(d_1)e^{-qT} - KN(d_2)e^{-rT} - S(0)e^{-qT}[H/S(0)]^{2\lambda}N(y)$$
$$+Ke^{-rT}[H/S(0)]^{2\lambda-2}N(y - \sigma\sqrt{T})$$

and that the price of an up-and-out call is

$$f_{\text{bs}} = S(0)e^{-qT}[N(d_1) - N(x_1)] - Ke^{-rT}[N(d_2) - N(x_1 - \sigma\sqrt{T})]$$
$$+S(0)e^{-qT}[H/S(0)]^{2\lambda}[N(-y) - N(-y_1)]$$
$$-Ke^{-rT}[H/S(0)]^{2\lambda-2}[N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})]$$

where

discussed in Broadie, Glasserman, and Kou (1997).

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$
$$y = \frac{\ln\{H^2/[S(0)K]\}}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$
$$x_1 = \frac{\ln[S(0)/H]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$
$$x_2 = \frac{\ln[H/S(0)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$
$$d_1 = \frac{\ln[S(0)/K] + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

and N is the cumulative normal distribution function. In computing f_{bs} we set σ equal to the implied volatility of a regular European call option with strike price K maturing at time T.

Tables 4 and 5 show f_{sv} and the percentage errors when the option is approximated by f_{ivf} and f_{bs} for the cases where K are 90% and 100% of the initial asset price and the time to maturity is two years. We consider a wide range of values for the barrier H. (When $H > 100$ the option is an up-and-out call; when $H < 100$ it is a down-and-out call.) A comparison of Table 3 with Tables 4 and 5 shows that the IVF model does not perform as well for barrier options as it does for compound options. For some values of the barrier, errors are high in both absolute terms and percentage terms in both the stock index and foreign currency cases. We carried out some tests (not reported here) using options with maturities other than two years. We found that the maximum potential percentage pricing error in the IVF model increases with the maturity of the option.

We conclude from these and other similar tests that there is significant model risk when the IVF model is used to price barrier options for some sets of parameter values. This is an important result. Barrier option are amongst the most popular exotic options.

V. Hedging Using the CR-IVF Model

As already mentioned, when traders use the CR-IVF model, they hedge against changes in the volatility surface as well as against changes in the asset price. The model has the form shown in equation (4). Traders calculate $\partial P/\partial c_i$ for $1 \leq i \leq n$ (or equivalent partial derivatives involving attributes of the volatility surface) as well as $\partial P/\partial S$ and attempt to combine their positions in exotic options with positions in the underlying asset and positions in European options to create a portfolio that is riskless when it is valued using the CR-IVF model.[13] They means that they create a portfolio whose value, Π, (as measured by the CR-IVF model) satisfies

$$\frac{\partial \Pi}{\partial c_i} = 0$$

for $(1 \leq i \leq n)$ and

$$\frac{\partial \Pi}{\partial S} = 0$$

Our test of the hedging effectiveness of the CR-IVF model is analogous to our test of its pricing effectiveness. We assume that the fitted two-factor stochastic volatility model in equations (5) and (6) gives the true price of an exotic option. This model has two underlying stochastic variables: S, the asset price and v, the volatility. We test whether that the CR-IVF model gives reasonable estimates of the sensitivities of the exotic option price changes to S and v.[14] We calculate numerically the partial derivative of exotic option prices with respect to each of S and v for both the stochastic volatility model and the CR-IVF model. The partial derivative with respect to S is the delta, Δ, and the partial derivative with respect to v is the vega, $\mathcal{V}$, of the exotic option.

[13] Given the nature of the CR-IVF model the partial derivatives of P with respect to S and the c_i cannot be calculated analytically. They must be calculated by perturbing the stock price and each of the n option prices in turn, recalibrating the model, and observing the effect on the exotic option price, P.

[14] This is an indirect way of testing whether a hedge portfolio consisting of the underlying asset and plain vanilla options will work.

In order to compute Δ, we increase the spot price by 1%, and compute the price changes from both the stochastic volatility model and the CR-IVF model.[15] To compute vega we increase the initial instantaneous volatility by 1% and compute price changes for both the stochastic volatility model and the CR-IVF model.[16] Note that the calculation of delta for the CR-IVF model involves calibrating the model twice to the option prices given by the stochastic volatility model, once before and once after S has been perturbed. The same is true of vega.

Our hedging results for the compound options considered in Table 3 are shown in Table 6. The table shows that using the CR-IVF model for hedging gives good results in a world described by the two-factor stochastic volatility model. Our hedging results for the barrier option considered in Table 4 are shown in Table 7. The percentage errors in the deltas calculated by the CR-IVF model are similar to the percentage pricing errors. The vegas calculated for the CR-IVF model are quite often markedly different from the vegas calculated using the stochastic volatility model. This is indication that volatility hedges created using the CR-IVF model for barrier options may not be effective. The tests for the barrier options in Table 5 and others we considered produced similar results.

[15] We also computed delta for the SC-IVF model. These are not reported, but are slightly worse than the deltas from the CR-IVF model.

[16] As already mentioned, in practice traders calculate vega by shifting the volatility surface. A number of vegas might be calculated, each corresponding to a different shift. Given our set up, which assumes that the stochastic volatility model is the true model and comparing results with that model, we look at only one vega.

VI. Economic Intuition for Results

Our results show that the CR-IVF model works quite well for compound options and much less well for barrier options. This is consistent with the observation in Section III that the payoff from a compound option is much less dependent on joint distribution of the asset price at different points in time than it is for barrier option.

Figure 1 provides another way of understanding the results. It shows the Black–Scholes price of the equity index barrier option as a function of volatility when the strike price is 100 and the barrier is 130. This has significant convexity. As shown by Hull and White (1987), when there is zero correlation between the asset price and volatility, the stochastic volatility price of a barrier option is its Black–Scholes price integrated over the probability distribution of the average variance rate during the life of the option. The convexity therefore leads to the value of the option increasing as the volatility of the volatility increases in the zero correlation case. For the equity index, the correlation between stock price and volatility is negative rather than zero. This increases the value of the option still further because high stock prices tend to be associated with low volatilities making it less likely that the barrier will be hit.

These arguments explain why the Black–Scholes price of the option we are considering is 38.61% less than the stochastic volatility price. Consider next the IVF price. The IVF model does incorporate a negative correlation between stock price and volatility. However, it is a one-factor model and does not lead to as wide a range of high and low volatility outcomes as the stochastic volatility model. As a result it does not fully reflect the impact of the convexity in Figure 1 and gives a lower price than the stochastic volatility model.

Other big discrepancies between the IVF price and the stochastic volatility price in Tables 4 and 5 can be explained similarly. For compound options, and for barrier options when the IVF percentage error is low, there is very little convexity in the relationship between the option price and volatility and as a result the IVF model works much better.

VII. Summary

It is important that tests of model risk reflect the way models are actually used by traders. Researchers and traders often use models differently. A model when it is developed by researchers is usually a SC (single calibration) model. When the same model is used by traders it is a CR (continual recalibration) model and is used to provide relative valuation among derivative securities. The researcher-developed SC models are usually arbitrage free. The corresponding CR models used by traders are not necessarily arbitrage free. This paper presents a methodology for testing the use of a CR model for valuing illiquid securities and applies the methodology to the implied volatility function (IVF) model.

The CR-IVF model has the attractive feature that it always matches the prices of European options. This means that the unconditional probability distribution of the underlying asset price at all future times is always correct. An exotic option, whose payoff is contingent on the asset price at just one time is always correctly priced by the CR-IVF model. Unfortunately, many exotic options depend in a complex way on the joint probability distribution of the asset price at two or more times. There is no guarantee that the CR-IVF model will provide good pricing and hedging for these instruments.

In this paper we examine the model risk in the CR-IVF model by fitting a stochastic volatility model to market data and then comparing the prices of compound options and barrier options with those given by the IVF model. We find that the CR-IVF model gives reasonably good results for compound options. The results for barrier options are much less satisfactory. The CR-IVF model does not recover enough aspects of the dynamic features of the asset price process to give reasonably accurate prices for some combinations of the strike price and barrier level. The hedge parameters produced by the model also sometimes have large errors.

An analysis of the CR-IVF model leads to the conjecture that the performance of the model should depend on the degree of path dependence in the exotic option being priced where "degree of path dependence" is defined as the number of times the asset price must be observed to calculate the payoff. The higher the degree of path dependence the worse the model is expected to perform. Compound options have a much lower degree of path dependence than barrier options. Our results are therefore consistent with the conjecture.

References

Andersen, L., and J. Andreasen. “Factor Dependence of Bermudan Swaptions: Fact or Fiction.” *Journal of Financial Economics*, 62, 1 (2001), 3–37.

Andersen, L., and R. Brotherton-Ratcliffe. “The Equity Option Volatility Smile: An Implicit Finite-Difference Approach.” *Journal of Computational Finance*, 1, 2 (1998), 5-37.

Bakshi, G., C. Cao, and Z. Chen. “Empirical Performance of Alternative Option Pricing Models.” *Journal of Finance*, 52, 5 (1997), 2003-49.

Basle Committee on Banking Supervision. *A New Capital Adequacy Framework.* Bank for International Settlements, Basle, 1999.

Bates, D. “Jumps and Stochastic Volatilities: Exchange Rate Processes Implicit in the Deutsche Mark Options.” *Review of Financial Studies*, 9 (1996), 69-107.

Black, F. and M. S. Scholes. “The Pricing of Options and Corporate Liabilities.” *Journal of Political Economy*, 81 (1973), 637-59.

Boyle, P. P. “Options: A Monte Carlo Approach.” *Journal of Financial Economics*, 4 (1977), 323-38.

Brace, A., B. Goldys, F. Klebaner, and R. Womersley. “Market Model of Stochastic Implied Volatility with Application to the BGM Model.” Working Paper S01-1, Department of Statistics, University of New South Wales, 2001.

Britten–Jones, M., and A. Neuberger. “Option Prices, Implied Price Processes, and Stochastic Volatility.” *Journal of Finance*, 55, 2 (2000), 839–66.

Broadie, M., P. Glassermann, and S. G. Kou. “A Continuity Correction for Discrete Barrier Options.” *Mathematical Finance*, 7 (1997), 325–49.

Buraschi, A., and J. Jackwerth. “The Price of a Smile: Hedging and Spanning in options Markets.” *Review of Financial Studies*, 14, 2 (2001), 495–528.

Derman, E. (1996) “Model risk.” *Quantitative Strategies Research Notes*, Goldman Sachs, New York, NY, 1996.

Derman, E., and I. Kani. (1994), “The Volatility Smile and its Implied Tree.” *Quantitative Strategies Research Notes*, Goldman Sachs, New York, NY, 1994.

Driessen, J., P. Klaassen, and B. Melenberg. “The Performance of Multi-Factor Term Structure Models for Pricing and Hedging Caps and Swaptions.” Working Paper, Department of Econometrics and CentER, Tilburg University, 2000.

Dumas, B., J. Fleming and R. E. Whaley. “Implied Volatility Functions: Empirical Tests.” *Journal of Finance* 53, 6 (1998), 2059–2106.

Dupire, B. “Pricing with a Smile.” *Risk*, 7 (1994), 18–20.

Geske, R. “The Valuation of Compound Options.” *Journal of Financial Economics*, 7, 1 (1979), 63-82.

Goldman, B., H. Sosin, and M. A. Gatto. “Path Dependent Options: Buy at the Low, Sell at the High.” *Journal of Finance*, 34, 5 (1979) 1111-27.

Green, T.C., and S. Figlewski. “Market Risk and Model Risk for a Financial Institution Writing Options.” *Journal of Finance*, 54, 4 (1999), 1465-99.

Gupta A., and M. G. Subrahmanyam. "An Examination of the Static and Dynamic Performance of Interest Rate Models in the Dollar Cap–Floor Markets." Working Paper, Weatherhead School of Management, Case Western Reserve University, 2000.

Heston, S. L. "A Closed-Form Solution for Options with Stochastic Volatility Applications to Bond and Currency Options." *Review of Financial Studies*, 6, 2 (1993), 327-43.

Hull, J. and A. White. "The Pricing of Options with Stochastic Volatility." *Journal of Finance*, 42, 2 (1987), 281-300.

Hull, J. and A. White. "An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility." *Advances in Futures and Options Research*, 3 (1988), 27-61.

Hull, J. and A. White. "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model." *Journal of Fixed Income*, 10, 2 (2000) 46–62.

Jackwerth, J. C. and M. Rubinstein. "Recovering Probability Distributions from Option Prices." *Journal of Finance*, 51, 5 (1996), 1611–31.

Ledoit, O. and P. Santa-Clara. "Relative Pricing of Options with Stochastic Volatility." Working Paper, Anderson Graduate School of Management, University of California, Los Angeles, 1998.

Longstaff, F. A., P. Santa-Clara, and E. Schwartz. "Throwing away a Billion Dollars: The Cost of Suboptimal Exercise Strategies in the Swaptions Market, *Journal of Financial Economics*, 62, 1 (2001), 39-66.

Melino, A. and S. M. Turnbull. "Misspecification and the Pricing and Hedging of Long-Term Foreign Currency Options." *Journal of International Money and Finance*, 14, 3 (1995), 373-93.

Merton, R. C. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science*, 4 (1973), 141-83.

Merton, R. C. "Option Pricing when Underlying Returns are Discontinuous." *Journal of Financial Economics*, 3 (1976), 125-44.

Rosenberg, J. "Implied Volatility Functions: A Reprise." *Journal of Derivatives*, 7 (2000), 51-64.

Rubinstein, M. "Implied Binomial Trees." *Journal of Finance*, 49, 3 (1994) 771-818.

Schöbel, R. and J. Zhu. "Stochastic Volatility with an Ornstein-Uhlenbeck Process: An Extension." Working Paper, Eberhard-Karls-Universität Tübingen, 1998.

Stein, E. and C. Stein. "Stock Price Distributions with Stochastic Volatilities: An Analytical Approach." *Review of Financial Studies*, 4, 4 (1991), 727-52.

Suo, W. (2001), "Modeling Implied Black–Scholes Volatility." Working Paper, School of Business, Queens University.

TABLE 1
Volatility Surface for Stock Index Options. K=strike price as a percent of the initial asset price; T=time to maturity.

	K											
T	70.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	130.00	140.00
0.175	29.67	28.06	27.31	26.58	25.87	25.16	24.47	23.79	23.13	22.48	21.29	21.35
0.425	29.71	28.20	27.47	26.76	26.07	25.39	24.71	24.05	23.40	22.77	21.56	20.49
0.695	29.82	28.34	27.64	26.95	26.27	25.61	24.95	24.31	23.68	23.06	21.88	20.81
0.940	29.91	28.47	27.78	27.11	26.45	25.80	25.17	24.54	23.93	23.33	22.17	21.11
1.000	29.94	28.50	27.82	27.15	26.49	25.85	25.22	24.60	23.99	23.39	22.24	21.18
1.500	30.11	28.74	28.08	27.45	26.82	26.21	25.61	25.02	24.45	23.88	22.78	21.76
2.000	30.27	28.95	28.32	27.71	27.12	26.54	25.97	25.41	24.86	24.32	23.28	22.30
3.000	30.53	29.31	28.73	28.18	27.63	27.10	26.58	26.08	25.58	25.09	24.15	23.26
4.000	30.74	29.61	29.08	28.56	28.06	27.57	27.10	26.63	26.18	25.74	24.89	24.07
5.000	30.91	29.86	29.36	28.88	28.42	27.97	27.53	27.11	26.69	26.29	25.51	24.76

The parameters for the stochastic volatility model used in generating this table are: $r = 5.9\%$, $q = 1.4\%$, $v_0 = 0.25$, $\kappa = 0.16$, $\theta = 0.3$, $\xi = 0.09$, and $\rho = -0.79$.

TABLE 2
Volatility Surface for Foreign Currency Options. K=strike price as a percent of the initial asset price; T=time to maturity.

	K											
T	70.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	130.00	140.00
0.175	23.03	14.56	12.85	12.73	12.75	12.83	12.96	13.13	13.32	13.56	16.16	21.10
0.425	14.97	12.90	12.77	12.72	12.74	12.81	12.93	13.09	13.27	13.46	13.88	14.50
0.695	13.44	12.89	12.77	12.71	12.72	12.79	12.90	13.06	13.23	13.42	13.81	14.20
0.940	13.32	12.89	12.76	12.70	12.71	12.77	12.88	13.02	13.19	13.37	13.76	14.14
1.000	13.31	12.89	12.76	12.70	12.70	12.76	12.87	13.02	13.18	13.36	13.74	14.12
1.500	13.30	12.89	12.76	12.69	12.68	12.73	12.82	12.96	13.11	13.29	13.65	14.02
2.000	13.30	12.89	12.76	12.69	12.67	12.70	12.78	12.90	13.05	13.21	13.56	13.92
3.000	13.30	12.91	12.77	12.68	12.65	12.66	12.72	12.81	12.94	13.08	13.40	13.73
4.000	13.31	12.92	12.78	12.69	12.64	12.63	12.67	12.74	12.84	12.97	13.25	13.56
5.000	13.31	12.93	12.80	12.70	12.64	12.62	12.63	12.69	12.77	12.87	13.13	13.41

The parameters for the stochastic volatility model used in generating this table are: $r = 5.9\%$, $q = 3.5\%$, $v_0 = 0.1285$, $\kappa = 0.1090$, $\theta = 0.10$, $\xi = 0.0376$, and $\rho = 0.1548$.

TABLE 3
Numerical Results for Compound Options (SV = Stochastic Volatility)

	Stock Index Data (Table 1)			Foreign Currency Data (Table 2)		
K_1	SV Price	IVF % Error	BS % Error	SV Price	IVF % Error	BS % Error
3.45	15.45	-0.05	0.43	6.24	-0.95	-0.72
5.90	13.60	-0.14	1.23	4.82	-1.64	-1.76
8.35	11.93	-0.25	2.47	3.70	-1.67	-3.13
10.80	10.42	-0.36	4.14	2.83	-1.73	-4.78
13.25	9.07	-0.45	6.28	2.14	-1.63	-6.77
15.70	7.85	-0.54	8.99	1.62	-1.42	-9.22
18.15	6.76	-0.53	12.31	1.21	-1.07	-12.13
20.60	5.79	-0.59	16.34	0.90	-0.77	-15.49
23.05	4.92	-0.47	21.14	0.67	-1.04	-19.49
25.50	4.16	-0.36	26.93	0.50	-0.80	-24.05
27.95	3.50	-0.26	33.77	0.37	0.81	-28.72
30.40	2.92	-0.14	41.89	0.27	1.47	-33.82
32.85	2.42	0.00	51.39	0.20	1.00	-39.00
35.30	1.99	0.25	62.58	0.15	2.74	-43.84
37.75	1.62	0.62	75.85	0.11	4.67	-49.53
40.20	1.32	1.06	91.48	0.08	5.13	-55.13
42.65	1.06	1.42	110.02	0.06	7.02	-59.65
45.10	0.84	1.68	131.75	0.04	7.14	-64.29

The table shows the stochastic volatility price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The maturity of the first option, T_1, is 1 year; the maturity of the second option, T_2, is 2 years; the second strike price, K_2 equals the initial asset price; the first strike price K_1 is shown in the table as a percentage of the initial asset price.

TABLE 4
Numerical Results for Knock-Out Barrier Options when the Strike Price is 90% of the Initial Asset Price; SV = Stochastic Volatility

	Stock Index Data (Table 1)			Foreign Currency Data (Table 2)		
H	SV Price	IVF % Error	BS % Error	SV Price	IVF % Error	BS % Error
50	24.00	0.09	0.18	14.94	0.07	0.02
60	23.77	-0.06	0.64	14.94	0.04	0.02
70	22.69	0.00	1.67	14.93	0.09	0.05
80	19.74	-0.42	0.75	14.72	0.33	0.37
90	12.97	-0.69	-3.44	12.08	-0.75	-0.57
92	10.95	-0.74	-4.67	10.58	-0.51	-0.09
94	8.72	-1.20	-6.54	8.67	-0.56	0.10
96	6.18	-1.84	-8.75	6.32	-1.32	-0.08
98	3.36	-4.08	-12.84	3.42	-1.44	0.50
104	0.03	-26.60	-17.49	0.26	-21.65	-26.61
106	0.06	-25.91	-20.02	0.52	-18.28	-22.92
108	0.11	-26.61	-23.54	0.87	-16.81	-20.16
110	0.18	-24.16	-24.84	1.32	-14.33	-17.43
120	1.01	-18.54	-31.40	4.42	-6.62	-4.37
130	2.90	-13.85	-36.74	7.74	-2.71	3.58
140	5.91	-10.39	-40.49	10.41	-1.37	6.40
150	9.66	-8.15	-42.35	12.19	-0.52	6.81

The table shows the stochastic volatility price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The barrier, H, is shown in the table as a percent of the initial asset price. The time to maturity is two years.

TABLE 5
Numerical Results for Knock-Out Barrier Options when the Strike Price Equals the Initial Asset Price.

	Stock Index Data (Table 1)			Foreign Currency Data (Table 2)		
H	SV Price	IVF % Error	BS % Error	SV Price	IVF % Error	BS % Error
50	18.40	0.07	0.13	8.93	0.01	0.02
60	18.29	-0.05	0.54	8.93	0.01	0.02
70	17.61	0.11	2.12	8.93	0.02	0.04
80	15.61	-0.25	2.79	8.86	0.39	0.42
90	10.55	-0.54	0.52	7.72	0.48	0.37
92	8.96	-0.59	-0.37	6.91	1.14	1.10
94	7.18	-0.97	-1.91	5.84	1.11	1.10
96	5.12	-1.64	-3.91	4.39	0.30	0.64
98	2.80	-3.90	-7.96	2.45	0.04	0.88
104	0.00	-42.41	-9.93	0.01	-46.41	-34.04
106	0.00	-36.50	-19.52	0.03	-33.51	-29.92
108	0.01	-40.69	-28.23	0.09	-30.36	-25.62
110	0.03	-32.00	-27.10	0.19	-24.08	-21.73
120	0.34	-24.21	-33.29	1.44	-10.56	-4.18
130	1.39	-18.08	-38.61	3.41	-4.36	6.64
140	3.40	-13.40	-42.20	5.27	-2.33	10.12
150	6.20	-10.18	-43.88	6.62	-0.90	10.38

The table shows the stochastic volatility price as a percent of the initial asset price, and the percentage error when this is approximated using the IVF model and the Black–Scholes model. The barrier, H, is shown in the table as a percent of the initial asset price. The time to maturity is two years.

TABLE 6
Price Sensitivities for Compound Options

	Stock Index Data (Table 1)				Foreign Currency Data (Table 2)			
K_1	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$
3.45	0.691	0.699	0.398	0.398	0.541	0.556	0.436	0.440
5.90	0.667	0.676	0.404	0.407	0.473	0.496	0.421	0.435
8.35	0.637	0.641	0.409	0.411	0.402	0.392	0.394	0.393
10.80	0.603	0.606	0.412	0.414	0.335	0.332	0.362	0.362
13.25	0.566	0.568	0.412	0.410	0.274	0.272	0.324	0.325
15.70	0.528	0.533	0.408	0.411	0.219	0.218	0.283	0.284
18.15	0.489	0.498	0.402	0.405	0.174	0.170	0.244	0.244
20.60	0.449	0.446	0.392	0.389	0.136	0.131	0.207	0.207
23.05	0.410	0.405	0.380	0.377	0.106	0.106	0.172	0.173
25.50	0.371	0.367	0.364	0.361	0.081	0.088	0.142	0.145
27.95	0.333	0.330	0.346	0.343	0.062	0.065	0.116	0.119
30.40	0.296	0.296	0.325	0.328	0.048	0.048	0.094	0.096
32.85	0.261	0.261	0.303	0.306	0.036	0.036	0.076	0.077
35.30	0.228	0.227	0.279	0.281	0.027	0.029	0.060	0.062
37.75	0.197	0.199	0.255	0.256	0.020	0.022	0.047	0.050
40.20	0.170	0.175	0.231	0.231	0.015	0.016	0.037	0.039
42.65	0.144	0.140	0.206	0.209	0.011	0.012	0.029	0.031
45.10	0.121	0.123	0.182	0.185	0.008	0.009	0.023	0.024

Δ and Δ_{ivf} are the rate of change of the option price with the asset price for the stochastic volatility model and the CR-IVF model, respectively. $\mathcal{V}$ and $\mathcal{V}_{\text{ivf}}$ are the change in the option price (measured as a percent of the initial stock price) for a 1% change in the initial instantaneous volatility for the stochastic volatility model and the CR-IVF model, respectively. The maturity of the first option, T_1, is 1 year; the maturity of the second option, T_2, is 2 years; the second strike price, K_2 equals the initial asset price; the first strike price K_1 is shown in the table as a percentage of the initial asset price.

TABLE 7
Price Sensitivities for Knock-Out Barrier Options when the Strike Price is 90% of the Initial Asset Price.

	Stock Index Data (Table 1)				Foreign Currency Data (Table 2)			
K_1	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$	Δ	Δ_{ivf}	$\mathcal{V}$	$\mathcal{V}_{\text{ivf}}$
50	0.786	0.788	0.322	0.318	0.781	0.782	0.298	0.299
60	0.793	0.799	0.254	0.283	0.781	0.785	0.297	0.299
70	0.836	0.836	0.143	0.189	0.783	0.785	0.293	0.297
80	0.931	0.941	-0.034	0.035	0.807	0.807	0.244	0.217
90	1.186	1.182	-0.064	-0.107	1.077	1.079	-0.213	-0.132
92	1.269	1.260	-0.194	-0.112	1.185	1.198	-0.200	-0.188
94	1.326	1.356	-0.158	-0.110	1.327	1.320	-0.187	-0.212
96	1.457	1.429	-0.180	-0.093	1.460	1.464	-0.198	-0.191
98	1.442	1.530	-0.073	-0.058	1.657	1.634	-0.094	-0.120
104	-0.007	-0.007	-0.003	-0.001	-0.061	-0.052	-0.047	-0.034
106	-0.011	-0.008	-0.002	-0.003	-0.080	-0.071	-0.090	-0.070
108	-0.015	-0.011	-0.009	-0.006	-0.095	-0.081	-0.150	-0.116
110	-0.022	-0.017	-0.015	-0.010	-0.100	-0.096	-0.212	-0.174
120	-0.070	-0.055	-0.084	-0.063	-0.039	-0.045	-0.502	-0.486
130	-0.116	-0.116	-0.233	-0.195	0.141	0.118	-0.586	-0.614
140	-0.164	-0.162	-0.452	-0.403	0.333	0.318	-0.514	-0.534
150	-0.148	-0.148	-0.044	-0.039	0.488	0.487	-0.327	-0.362

Δ and Δ_{ivf} are the rate of change of the option price with the asset price for the stochastic volatility model and the CR-IVF model, respectively. $\mathcal{V}$ and $\mathcal{V}_{\text{ivf}}$ are the change in the option price (measured as a percent of the initial stock price) for a 1% change in the initial instantaneous volatility for the stochastic volatility model and the CR-IVF model, respectively. The barrier, H, is shown as a percent of the initial asset price. The time to maturity is two years.