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The Risk of Tranches Created from Mortgages

John Hull and Alan White

Joseph L. Rotman School of Management
University of Toronto

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Abstract

Using the criteria of the rating agencies, this paper tests how wide the AAA tranches created from residential mortgages can be. It concludes that the AAA ratings assigned to ABSs were not totally unreasonable, but the AAA ratings assigned to tranches of Mezz ABS CDOs cannot be justified.

The Risk of Tranches Created from Mortgages

The rating agencies have come under a great deal of criticism since the subprime crisis started in July 2007. It is argued that the AAA ratings assigned to the structured products created from residential mortgages convinced investors that the products were almost completely free of risk. As a result, the investors were lulled into a false sense of confidence and did not evaluate the products for themselves. Recent research by Adelino (2009) supports this view. He tested whether yield spreads on the AAA-rated products at issuance contained information, in addition to that in their ratings, that would be useful in predicting subsequent performance. His conclusion was that they did not.[1]

The traditional business of rating agencies is the rating of corporate and sovereign bonds. This is based on a combination of analysis and judgment. The rating of structured products was a departure from this traditional business. Instead of analysis and judgment, it involved the application of a model. The rating agencies were quite open about the models they used. Standard and Poor's (S&P) and Fitch based their ratings on the probability of loss given by their models. If the calculated probability of loss on a structured product corresponded to the probability of loss on a AAA-rated bond, the structured product was rated AAA. Moody's, by contrast, based its ratings on the expected loss as a percent of principal. If the expected loss on a structured product corresponded to the expected loss on a Aaa-rated bond, the structured product was rated Aaa. When a structured product was designed, creators wanted to achieve their target ratings for tranches by meeting the model requirements of rating agencies.[2] Their objective was usually to make the total principal of the AAA tranches that they created as large as possible. Often, they got advance rulings on ratings before finalizing product design.

In this paper we evaluate whether the AAA ratings assigned to structured products by rating agencies were reasonable. We look at both asset-backed securities (ABSs), which were products created from pools of mortgages, and ABS collateralized debt obligations (ABS CDOs), which were created from the tranches of several ABSs. Coval *et al* (2008) also evaluate ratings for these types of products. However, they assume that the asset pools underlying ABS CDOs have zero default correlation with each other. We do not make this assumption. Indeed, we find that

[1] Interestingly, the yield spreads did improve predictions for products with ratings below AAA.
[2] See Brennan, Hein, and Poon (2008) for a discussion of this.

the risks of the tranches in an ABS CDO are critically dependent on the correlation between different asset pools.

The Products That Were Created

During the 1999 to 2007 period, financial institutions found securitization increasingly attractive. There were a number of reasons for this. Securitization was a way of reducing regulatory capital. It was profitable because the weighted average interest paid on the securitized products was less than the weighted average interest earned on the underlying assets. (This is referred to as "spread arbitrage"). It was an essential aspect of the "originate-to-distribute" model that was used by many banks. Rather than keeping assets they originated on their balance sheets, the credit risk could be passed on to investors. Many different types of assets were securitized: corporate debt, credit card receivables, car loans, etc. In this paper we focus on the securitization of subprime residential mortgages.

ABSs

The nature of the ABSs that were created from subprime residential mortgages is discussed in some detail by Gorton (2007). A typical structure is illustrated in Figure 1. The subprime mortgage portfolio might consist of a total of 1000 mortgages. This underlying collateral is allocated to one or more senior tranches rated AAA, a number of mezzanine tranches rated AA, A, and BBB, and subordinated tranches which are either rated BB or unrated. Principal is allocated to each tranche. Sometimes, the total principal of the mortgages equals the total principal of the tranches. In other cases, there is some overcollateralization where the total principal of the mortgage portfolio exceeds the total principal of the tranches.

A key aspect of the design of the structure is the amount of principal allocated to each rating category. Typically 75% to 85% of the mortgage principal was allocated to AAA tranches. The principals allocated to other rating categories were much smaller. For example, the BBB tranches taken together typically accounted for 3% or less of the mortgage principal. One of the goals of the creator of the ABS was to create value from spread arbitrage, as mentioned above. The greatest value is created when the credit quality (as measured by the credit rating) of the tranches is maximized. First, the AAA tranches were made as wide as possible; the AA tranches were then made as wide as possible; and so on.

The rules for allocating cash flows from the mortgages to tranches are defined by what is known as a "waterfall." The interest payments promised to tranches typically decrease with seniority. Interest payments from mortgages are typically allocated to tranches in order of seniority so that the AAA-rated tranches get promised interest payments on their outstanding principal first; after that, the AA-rated tranches get their promised interest payments on their outstanding principal; and so on.

The principal payments (both scheduled and prepayments) are handled separately from interest payments and the rules are relatively complicated. There is typically a lockout period during which principal payments are sequential. This means that all principal payments go first to the most senior tranche. When that tranche has been completely amortized, they go to the next-most-senior tranche, and so on. After the lock out period, if certain performance targets are met, principal payments are allocated to tranches in proportion to their outstanding principals. However, if there is a "cumulative loss trigger event" (where cumulative losses on the mortgages are higher than certain levels) or a "delinquency event" (where the rate of delinquency over a three-month period is above a certain level), principal payments become sequential again.

As an approximation, it can be assumed that, when the default rate proves to be high, the repayment of principal is entirely sequential. The effect of this is that tranches bear losses in order of reverse seniority. The unrated tranches absorb losses first. Once their principal has been lost, the BB-rated tranches bear losses, and so on.

There were usually several tranches corresponding to each rating category. For example, in the Structured Asset Investment Loan Trust (SAIL), issued in 2006, there were four AAA tranches (with equal seniority) accounting for 83.25% of the collateral; two AA-rated tranches (with unequal seniority) accounting for 8.2% of the collateral; three A-rated tranches (with unequal seniority) accounting for 4.1% of the principal; and three BBB tranches (with unequal seniority) accounting for 2.8% of the principal.

The BBB tranches were often very thin. Although the total of all BBB tranches might account for about 3% of the total underlying principal, each individual BBB tranche was often only about

1% wide. If the macroeconomic environment is relatively benign so that there are large repayments of principal, the AAA tranche can be expected to shrink and the proportion of the remaining mortgage principal accounted for by the BBB tranches can be expected to increase. But, if default rates are high, a thin BBB tranche can easily lose its entire principal. In the SAIL structure just mentioned, the most senior BBB tranche (rated Baa1/BBB+) was 1.1% wide with 3.25% subordination; the next BBB tranche (rated Baa2/BBB) was 0.85% wide with 2.40% subordination, and the most junior BBB tranche (rated Baa3/BBB-) tranche was 0.8% wide with 1.60% subordination. Assuming principal payments are sequential, the three tranches will lose their entire principal if losses on the subprime mortgage portfolio are greater than 4.35%, 3.25%, and 2.40%, respectively.

ABS CDOs

In the second level of securitization, tranches are created from tranches. The products created are known as ABS CDOs. Two types of ABS CDOs were common. These were a "High Grade ABS CDO" created from the AAA, AA, and A tranches of ABSs and a "Mezz ABS CDO" created from the BBB tranches of ABSs.

We will focus on the Mezz ABS CDO. Its creation is illustrated in Figure 2. The AAA-rated tranche in Figure 2 is typically less wide than that in Figure 1. However, it still usually accounted for more than 50% of the ABS CDO principal. (In an example given by Gorton on page 35, which is taken from a UBS publication, the AAA-rated tranche of the ABS CDO accounts for 76% of the principal.)

Many ABS CDOs are managed. This means that the tranches forming the collateral do not remain fixed over time. A portfolio manager is allowed to trade a certain percentage of the underlying collateral each year. However there are restrictions relating to measures involving the ratings of the collateral, correlation, and the weighted average life of the underlying assets.

ABS CDOs are like ABSs in that the waterfall is complex. Losses tend to be allocated to the most junior tranches first. There are coverage tests and triggers which cause amortizations to be sequential and divert cash flows from junior to senior tranches. In certain circumstances, the senior tranche holders may be able to liquidate the assets

The Default Model

This paper focuses on the criteria used by the rating agencies. It tests, under a variety of different assumptions, a) what the attachment point for a AAA-rated tranche should be if it is to have the same probability of experiencing losses as a AAA-rated corporate bond and b) what the attachment point should be if it is to have the same expected loss of principal as a AAA-rated corporate bond. There are three components to the model:

1. An estimate of the expected default rate (EDR) for the mortgages in the underlying portfolio (i.e., an estimate of the expected proportion of the mortgages in the portfolio that will default)
2. A correlation model that converts the expected default rate to a probability distribution for the actual default rate
3. A specification of the expected loss given default (ELGD) as a percentage of the initial mortgage principal

Our model is relatively simple in that it does not incorporate a model of prepayment behavior and the timing of defaults. A more elaborate model would be essential for valuation. However, our objective is to test the reasonableness of what rating agencies did, not to value the securities. The rating agency criteria do not consider the timing of defaults (although arguably they should).[3] Modeling prepayments explicitly would be more accurate, but involve a big increase in the complexity of our model. We assume that EDR and ELGD estimates incorporate the effect of prepayments.

We do not explicitly consider overcollateralization. In high-default-rate situations, *x*% of overcollateralization can be thought of as a "dummy" junior tranche that absorbs the first *x*% of losses. The attachment points we report reflect the total subordination including overcollateralization.

[3] The timing of defaults is particularly important for the valuation of lower rated tranches because interest payments form a larger component of the return for these tranches.

We assume that principal payments are allocated to tranches sequentially so that losses are allocated in reverse order of seniority.[4] As mentioned earlier, this corresponds to the way ABS CDOs usually work. It also corresponds to the way ABSs usually work for the first few years and to the way they usually work in subsequent years when the default rate is high. In assigning ratings we are interested in observing how tranches fare in high-default-rate situations. The assumption that principal is always allocated sequentially is therefore reasonable for ABSs as well as ABS CDOs.

The mortgages in the pool are assumed to have equal principal and to have the same probability of default. A mortgage pool is assumed to be sufficiently large that a "large portfolio assumption" applies so that the actual proportion of mortgages defaulting in the portfolio equals the probability of each mortgage defaulting. (We refer to this as the "default rate.") In practice, there are about 1000 mortgages in a pool. Tests we have carried out show that the large portfolio assumption (which considerably reduces computation time) has only a small effect on our results.

Single Pool Correlation Model

Suppose that Q is the fraction of original mortgages in the pool that are expected to default within T years. If all the mortgages are of similar risk then Q is the probability of default for any individual mortgage. A natural model to assume is the one-factor Gaussian copula model. This was originally suggested by Li (2000) and, as a result of research by Gregory and Laurent (2005) and others, has become the standard market model for valuing synthetic CDOs. In this model, there is a factor that is common to all mortgages, which we will denote by M, and a factor specific to mortgage i which we will denote by Z_i. The factors M and Z_i are assumed to have independent standard normal distributions. In the model, mortgage i defaults within T years if

$$\sqrt{\rho}M + \sqrt{1-\rho}Z_i < K$$

For some K where ρ is the correlation between the transformed times to default of any two mortgages. Under the assumptions of standard normal distributions the probability of default is

[4] We do not consider the allocation of interest because the rating agency models are concerned only with the impairment of principal.

$N(K)$ where N is the cumulative normal distribution function. The model is calibrated to the expected default rate by setting $N(K) = Q$.

The ith mortgage therefore defaults if

$$\sqrt{\rho}M + \sqrt{1-\rho}Z_i < N^{-1}(Q)$$

or

$$Z_i < \frac{N^{-1}(Q) - \sqrt{\rho}M}{\sqrt{1-\rho}}$$

The realized default rate, P, conditional on M is therefore

$$P = N\left(\frac{N^{-1}(Q) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{1}$$

Hull and White (2004) show that any zero mean unit variance distributions can be chosen for M and Z_i. They find that the "double t" copula model where both M and Z_i have t-distributions with 4 degrees of freedom (scaled so that the variance is one) fits market data on synthetic CDOs well. It has considerably more tail default correlation (i.e., it has a higher probability of extreme clustering of defaults) than the Gaussian copula model.

In the double t copula model, the ith mortgage defaults if

$$\sqrt{\rho}M + \sqrt{1-\rho}Z_i < F^{-1}(Q)$$

where F is the cumulative probability distribution of:[5]

$$\sqrt{\rho}M + \sqrt{1-\rho}Z_i$$

The realized default rate, conditional on the factor M, is

$$P = H\left(\frac{F^{-1}(Q) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{2}$$

[5] In general, this distribution has to be determined numerically.

where H is the cumulative probability distribution of a scaled t-distribution with four degrees of freedom.

We will present results for tests assuming both the Gaussian copula model and the double t copula model.

The Multi-Pool Correlation Model

When several pools are considered simultaneously it is necessary define a "between-pool" factor, M_{bp}, and "within-pool" factors, $M_{wp,j}$. The factor M_{bp} affects the probability of default for all mortgages while $M_{wp,j}$ affects the probability of default only for mortgages in pool j. In this model the ith mortgage in the jth pool defaults if

$$\sqrt{\alpha\rho}M_{bp} + \sqrt{(1-\alpha)\rho}M_{wp,j} + \sqrt{1-\rho}Z_{ij} < \Psi^{-1}(Q)$$

where Z_{ij} is a variable affecting only the ith mortgage in the jth pool and Ψ is the cumulative probability distribution of

$$\sqrt{\alpha\rho}M_{bp} + \sqrt{(1-\alpha)\rho}M_{wp,j} + \sqrt{1-\rho}Z_{ij}$$

The factors and the variables Z_{ij} are independent of each other.

As before, the parameter ρ is the total within pool correlation. The parameter α indicates the proportion of the default correlation that comes from a factor common to all pools. When $\alpha = 0$ the default rates of different pools are independent of each other. (As noted earlier, when the variables are normally distributed this is the model assumed by Coval *et al* (2008).) At the other extreme, when $\alpha=1$, there is a single factor affecting all mortgage defaults and the default rates in all mortgage pools are the same.

A two-factor model is important when ABS CDOs are considered. One of the advantages cited for ABS CDOs over ABSs is that investors benefit from the across-pool diversification as well as within-pool diversification. Suppose that half of the underlying pools of an ABS CDO consist entirely of mortgages on Florida homes while the other half consist entirely of mortgages on California homes. If the default rate in California is less than perfectly correlated with the default rate in Florida, there is a diversification benefit to investors. The parameter α measures this diversification benefit. If α is low this extra diversification is valuable to investors, but if α is high it has very little value. Research suggests that correlations increase in stressed market conditions. For example, Servigny and Renault (2002), who look at historical data on defaults and ratings transitions to estimate default correlations, find that the correlations are higher in recessions than in expansion periods. Das, Freed, Geng and Kapadia (2004) employ a reduced form approach and compute the correlation between default intensities. They conclude that default correlations increase when default rates are high. Ang and Chen (2002) find that the correlation between equity returns is higher during a market downturn. Given that they are most interested in what happens during stressed market conditions, this research suggests that rating agencies should have used a relatively high value of α.[6] It should be noted that if ABS mortgage pools are already well diversified across the United States, so that there is very little extra diversification benefit from forming an ABS CDO, then α should be close to 1.

The realized default rate for pool j conditional on M_{bp} and $M_{wp,j}$ is

$$\Phi\left(\frac{\Psi^{-1}(Q)-\sqrt{\alpha\rho}M_{bp}-\sqrt{(1-\alpha)\rho}M_{wp,j}}{\sqrt{1-\rho}}\right) \tag{3}$$

where Φ the cumulative probability distribution of Z_{ij}. The simplest version of the model is the case in which the M's and Z's have standard normal distributions. We will also consider the case where they all have t distribution with four degrees of freedom (scaled so that the variance is one). We refer to this as the "triple t copula model."

Recovery Rate Model

We define the recovery as the amount recovered in the event of a default as a percentage of the initial principal when there is a default. It is one minus the ELGD defined earlier. Credit derivatives models often assume that the recovery rate realized when there is a default is constant. This is less than ideal. As the default rate increases, the recovery rate for a particular

[6] The copula model could be modified to make correlation parameters dependent on the default rate. This was suggested by Andersen and Sidenius (2004).

asset class can be expected to decline. This is because a high default rate leads to more of the assets coming on the market and a reduction in price.[7]

As is now well known, this argument is particularly true for residential mortgages. In a normal market, a recovery rate of about 75% is often assumed for this asset class. If this is assumed to be the recovery rate in all situations, the worst possible loss on a portfolio of residential mortgages given by the model would be 25%, and the 25% to 100% senior tranche of an ABS created from the mortgages could reasonably be assumed to be safe. (In fact, recovery rates on mortgages have declined sharply in the high default rate environment experienced since 2007.)

Define the recovery rate when the default rate equals the expected default rate as R^*, the maximum recovery rate (occurring when the default rate is very low) as R_{max} and the minimum recovery rate (occurring when the default rate is very high) as R_{min}. We use the following simple recovery rate model for the actual recovery rate R[8]

$$R = R_{min} + \left(R_{max} - R_{min}\right)\exp\left(-aP\right) \tag{4}$$

where

$$a = -\frac{\ln\left[\left(R^* - R_{min}\right)/\left(R_{max} - R_{min}\right)\right]}{Q} \tag{5}$$

As before, P is the actual default rate and Q is the expected default rate. As P increases from zero to 100%, the recovery rate decreases from R_{max} to close to R_{min} in such as way that, when $P = Q$, $R = R^*$.[9] Using equation (1) or (2), R can be expressed as a function of M. The model is illustrated in Figure 3.

[7]The negative relationship between recovery rates and default rates has been documented for bonds by Altman *et al* (2005) and Moody's Investors Service (2008).

[8] Tests we have carried out show that our results are not very sensitive to the choice of the recovery rate model.

[9] For convenience, we will refer to R^*, the recovery rate observed when the realized default rate equals the expected default rate, as the average recovery rate and the loss rate associated with it as the average loss rate. However, this is not the mathematical expected recovery rate.

Subprime Default Experience

Subprime first mortgages became common in the United States in 1999. This means that in 2006 and 2007 rating agencies had relatively little experience of the performance of these mortgages.

Figure 4 shows statistics collected by Moody's in March 2007.[10] The charts show, for subprime mortgages originated in a certain year, the cumulative percentage that was "delinquent" after a certain number of months. For this purpose, delinquent mortgages are defined as the total of those where payments are more than 60 days overdue, those in foreclosure, and those where the properties are being sold by the lender. Moody's had over five years of experience for mortgages originated between 1999 and 2003. The cumulative default rate for mortgages originated some time ago was between 2% and 4%. Note that the percentage of delinquent loans in the charts does not increase monotonically with time. This is because borrowers who become delinquent sometimes subsequently catch up on their late payments, refinance, or sell the house.

Figure 4 shows that there were signs that mortgages originated in 2006 were performing worse than mortgages originated in the four previous years (first chart). However, in March 2007 they appeared to be performing similarly to mortgages originated between 1999 and 2001 (second chart). The percentage of mortgages in the delinquent category after 11 months for the 1999, 2000, and 2001 vintages mortgages were 6.10%, 7.63%, and 7.15%, respectively. The percentage for the 2006 vintage was similar.[11]

In March 2007, investors in the AAA tranches of ABSs could draw some comfort from the AAA ABX indices which indicated no serious impairment. The TABX index, which aims to track the value of AAA tranches formed from the BBB (BBB–) tranches of ABSs, stood at 92.75 (84.00) at the end of March 2007.

Of course, there were a number of warning signals. The S&P Case-Shiller Composite 10 house price index, which was set at 100 in January 2000, reached over 225 in mid-2006, but had started to decline by the beginning of 2007. Although few people anticipated the full extent of the fall in house prices that took place in over the next two years, there was general agreement that some

[10] See Moody's Investors Services (2007).

[11] However, the 11-month percentage calculated in March 2007 reflects only loans originated early in 2006. The percentage of all loans originated in 2006 that became delinquent loans after 11 months (calculated at the end of 2007) was 12.13%

decline would take place. For obvious reasons, home owners are much more likely to default when house prices are falling than when they are rising. Mortgage default experience during the 1999 to 2006 period should therefore have been treated with caution.

The evaluation of ABSs depends on a) the expected default rate, Q, for mortgages in the underlying pool, b) the default correlation, ρ, for mortgages in the pool, and c) the recovery rate, R. Data from the 1999 to 2006 period suggest a value of Q less than 5% assuming an average mortgage life of 5 years. But, as has been mentioned, a different macroeconomic environment could be anticipated over the next few years. It would seem to be more prudent to use an estimate of 10%, or even higher. We will present results for values of Q equal to 5%, 10%, and 20%. The Basel II capital requirements are based on a copula correlation of 0.15 for residential mortgages.[12] We will present results for values of ρ between 0.05 and 0.30. As already mentioned, a recovery rate of 75% is often assumed for residential mortgages, but this is probably optimistic in a high default rate environment. We will present results for the situation where the recovery rate is fixed at 75% and for the situation where the recovery rate model in the previous section is used with R^*=75%, $R_{\min}$=50% and $R_{\max}$=100%.

ABS CDOs also depend on the parameter, α. Loosely speaking, this measures the proportion of the default correlation that comes from a factor common to all pools. A value of α close to zero indicates that investors obtain good diversification benefits from the ABS CDO structure. In adverse market conditions some mezzanine tranches can be expected to suffer 100% losses while others incur no losses. However, a value of α close to one indicates that all mezzanine tranches will tend to sink or swim together. We do not know what estimates rating agencies made for α. (Ex post of course, we know that it was high.) We will therefore present results based on a wide range of values for this parameter.

[12] See Bank for International Settlements (2006, p77) and Hull (2009). Basel II uses essentially the same copula model that we do with M and the Z_i normally distributed.

Results

Although mortgages are amortized over many years, prepayments lead to a weighted average life of about five years. When determining the ratings of instruments created from mortgages, their losses are therefore compared with the losses on bonds over a five year period. Statistics published by Moody's for the period 1970 to 2007 show that the cumulative five-year probability of default for AAA and BBB bonds are as shown in Table 1. The expected loss in the table is calculated from the probability of default assuming a recovery rate of 40% (which is a typical recovery rate for a corporate bond).

The Probability of Loss Criterion for ABSs

Suppose that the attachment point for the AAA tranche of an ABS is X% so that the tranche is responsible for losses between X% and 100%. The probability of the tranche experiencing losses is the probability that losses on the underlying portfolio are greater than X%. Given our large portfolio assumption that the proportion of mortgages defaulting equals the default rate, the tranche experiences losses when the default rate is greater than

$$\frac{X}{1-R}$$

where R is the recovery rate on the mortgages. Equation (1) shows that this happens in the case of the Gaussian copula model when

$$(1-R)N\left(\frac{N^{-1}(Q)-\sqrt{\rho}M}{\sqrt{1-\rho}}\right) > X$$

From Table 1, the minimum attachment point is the value of X for which the probability of this is 0.1%. It follows that the minimum attachment point is

$$(1-R)N\left(\frac{N^{-1}(Q)-\sqrt{\rho}N^{-1}(0.001)}{\sqrt{1-\rho}}\right) \tag{6}$$

The variable R is the recovery rate when $M = N^{-1}(0.001)$.

Similarly, equation (2) shows that for the double *t* copula model the minimum attachment point is

$$(1-R)H\left(\frac{F^{-1}(Q)-\sqrt{\rho}H^{-1}(0.001)}{\sqrt{1-\rho}}\right)$$

where, as before, H is the cumulative probability distribution for a *t*-distribution with four degrees of freedom (scaled so that the variance is one). In this case, R is the recovery rate when the $M = H^{-1}(0.001)$.

Table 2 shows results for different values of the expected default rate, Q, and the copula correlation, ρ. Four different models are considered:

i. The Gaussian copula model with a recovery rate of 75% on the underlying mortgages
ii. The double *t*-copula with a recovery rate of 75% on the underlying mortgages
iii. The Gaussian copula model with the stochastic recovery rate model in equations (4) and (5) with R^*=75%, R_{max}=100% and R_{min}=50%
iv. The double *t* copula model with the stochastic recovery rate model in equations (4) and (5) with R^*=75%, R_{max}=100% and R_{min}=50%

As might be expected, the minimum attachment point increases as we move from the Gaussian copula to the double *t*-copula and from the constant recovery rate model to the stochastic recovery rate model. As mentioned, the attachment point for AAA-rated tranches was typically 15% to 25%. There are some indications that attachment points were raised in 2006. To quote from Moody's Investment Services (2007) "Moody's Aaa-rated bonds issued in 2006 were designed to withstand a total loss on the underlying mortgage pool of approximately 26% to 30% without defaulting."

Table 2 shows that, when a 20% default rate is combined with a high default correlation and a stochastic recovery rate model, the AAA ratings that were made seem a little high. Also, the ratings are difficult to justify when the most extreme model (double t copula, stochastic recovery rate) is used. But overall the results in Table 2 indicate that the AAA ratings that were assigned were not totally unreasonable, given the published criteria of rating agencies.

The Expected Loss Criterion for ABSs

If $L(M)$ is the proportional loss on the mortgage portfolio for a particular value of M, the expected proportional loss on the ABS when the attachment point for the senior tranche is X is

$$\int_{M^*}^{\infty}\left[L(M)-X\right]\theta(M)\,dM \tag{7}$$

where M^* is the value of M that leads to a loss on the portfolio equal to X and θ is the probability density of M. Because $L(M)$ is always less than $1-R_{\min}$, $L(M)-X$ is also less than $1-R_{\min}$. It follows that the expected loss is always less than $1-R_{\min}$ times the probability of a loss. Assuming that $R_{\min}$, the minimum recovery rate on mortgages, is greater than the recovery rate assumed on bonds, it follows that a value of X that satisfies the probability of loss criterion must also satisfy the expected loss criterion.

To put this another way, the minimum attachment point when the expected loss criterion is used must be less than the minimum attachment point when the probability of loss criterion is used.[13] This is confirmed by Table 3 for the case where the model is a double *t* copula with stochastic recovery. It can be seen that, even when this exacting model is used, the expected loss criterion would lead to a 70% to 75% wide AAA-rated senior tranche being judged to be reasonable when $\rho = 0.1$.

The expected loss from a tranche equals the probability of loss multiplied by the expected loss given default. The expected loss given default is typically quite low for the most senior tranche. This means that expected loss is relatively low for this tranche and explains why it relatively easy to get a AAA rating when the expected loss measure is used. For more junior tranches, which tend to be quite thin, expected loss given default is quite high. (In the limit as a tranche become infinitesimally thin the expected loss given default is one.) This means that expected loss is relatively high for these tranches and tends to produce more conservative ratings than probability of loss.[14]

[13] This is discussed further in Das and Stein(2009)

[14] For a discussion of this see Moody's Investors Service (2007)

The Creation of BBB Tranches

BBB tranches must usually satisfy both the Moody's and S&P/Fitch criteria. Interestingly, the S&P/Fitch criterion depends only on the attachment point whereas the Moody's criterion depends on both the attachment point and the tranche width. It is likely that in practice the minimum attachment point was determined using the S&P/Fitch criterion and the minimum tranche width was determined using the Moody's criterion.

As an example of how this might work, suppose that the Gaussian copula model with a constant recovery rate is used. Suppose further that the expected default rate, copula correlation and recovery rate are 7%, 0.1, and 75%. The minimum attachment point is the attachment point that gives 1.8% in equation (6). This is found to be 4.90%. The expected loss in equation (7) can be calculated numerically. When the attachment point is 4.90%, the minimum detachment point is the detachment point that gives 1.08% in equation (7). Numerical analysis reveals that this is 5.93%. A 4.90% to 5.93% tranche therefore just satisfies the criteria of all three rating agencies. This type of analysis perhaps explains why BBB tranches were so thin.

Data in Stanton and Wallace (2008) and other data obtained by browsing the SEC web site suggests that the average subordination of BBB tranches created in 2006 was about 4% and the average tranche width was about 1%. In what follows the benchmark ABS CDO that we consider is therefore one where the underlying BBB tranches are responsible for losses on the underlying mortgage portfolio between 4% and 5%.

The Probability of Loss Criterion for ABS CDOs

The probability distribution of losses for an ABS CDO can be determined using Monte Carlo simulation.[15] Values for M_{bp} and $M_{bp,j}$ are simulated to determine the default rate and the loss rate for the mortgages in each pool. If the average loss rate is less than the attachment point, the loss on the ABS CDO tranche is zero. If it is greater than the detachment point, the loss on the ABS CDO is 100%. When the average loss rate is between the attachment point and the detachment point, there is a partial loss on the ABS CDO tranche.

[15] We find that the following analytic approximate approach gives good results. Calculate the mean and standard deviation of the loss on one BBB tranche of an ABS conditional on M_{bp}. Use the central limit theorem to estimate the conditional probability distribution of the average loss across all tranches. Integrate over M_{bp} to calculate the unconditional distribution.

We have produced results for the situation where the ABS CDO is created from 100 BBB tranches of CDSs, each tranche being responsible for losses in the range 4% to 5% of the underlying portfolio.[16] A number of different values for the α and ρ parameters are considered. The expected default rates of 5% and 10% on the underlying mortgages are considered. Analogously to before, the models we considered are:

i. The two-factor Gaussian copula model with a recovery rate of 75% on the underlying mortgages
ii. The two-factor triple *t*-copula with a recovery rate of 75% on the underlying mortgages
iii. The two-factor Gaussian copula model with the stochastic recovery rate model in equations (4) and (5) with R^*=75%, R_{max}=100% and R_{min}=50%
iv. The two-factor triple *t* copula model with the stochastic recovery rate model in equations (4) and (5) with R^*=75%, R_{max}=100% and R_{min}=50%

Table 4 presents results for (i) and (iv). As expected, the results for (ii) and (iii) are between these two extreme cases.

The pattern of results in Table 4 is different from that in Table 2. It is clear that the attachment point must be quite high for a wide range of assumptions In some cases the attachment point is so high that a AAA-rating for even a very thin senior tranche is not warranted (i.e., the minimum attachment point is 100%).

Tables 5 and 6 explore the impact of increasing the width of the underlying BBB tranches. In Table 5 all the tranches are responsible for losses between 4% and 7%. In Table 6 all the tranches are responsible for losses between 4% and 9%. The minimum attachment point does decrease as the tranche is made wider, but in all cases when one moves away from a low-α Gaussian copula model an attachment point below 50% becomes difficult to justify.

In practice, there is some heterogeneity in the underlying BBB tranches. Table 7 tests the effect of this by considering the situation where the attachment point has a uniform distribution between 2% and 6% and the tranche width has a uniform distribution (independent of the first

[16] Finding the AAA tranche attachment point is equivalent to determining the value at risk for a portfolio. In both cases we are seeking the level of loss that is exceeded only 0.1% of the time. Our estimates are based on 2.5 million simulations. The standard errors are fairly small, usually less than 0.5%.

uniform distribution) between 1% and 5%. The results show that the homogeneity assumption for the BBB tranches is not driving the results.

It should be noted that a CDO created from the triple BBB tranches of ABSs is quite different from a CDO created from BBB bonds. This is true even when the BBB tranches have been chosen so that their probabilities of default and expected losses are consistent with their BBB rating. The reason is that the probability distribution of the loss from a BBB tranche is quite different from the probability distribution of the loss from a BBB bond.

An insight into the characteristics of the loss distribution of BBB-rated tranches can be obtained by considering an extreme case. Suppose tranches are infinitesimally thin and $\alpha=1$ so that the losses on tranches are perfectly correlated with each other. It is then the case that either a) the BBB tranches lose none of their principal or b) each BBB tranche loses its entire principal. An ABS CDO consisting of a portfolio of these tranches suffers either zero loss or 100% loss. It follows that every tranche of the ABS CDO are also in the situation where they either lose everything or nothing. There means that there should be no differences between the ratings of the tranches. (Indeed, they should all be rated BBB.)

As explained earlier the BBB tranches that were created were often very thin. Furthermore, inspecting publicly available data on ABSs we find that the underlying mortgages are often from various parts of the United States rather than being concentrated in one geographical area, suggesting that α is quite high.

The Expected Loss Criterion for ABS CDOs

In the case of the senior ABS tranche, it was possible to show theoretically that the expected loss criterion always leads to lower minimum attachment points than the probability of loss criterion. We have not been able to produce a similar theoretical result for the senior ABS CDO tranche. However, our numerical results indicate that this is true in all the cases we have considered.

Conclusions

Contrary to many of the opinions that have been expressed, the AAA ratings for the senior tranches of ABSs were not totally unreasonable. The weighted average life of mortgages is about five years. For many of the assumptions that rating agencies might reasonably have made the probability of loss and expected loss of the AAA-rated tranches that were created were not markedly different from those of AAA-rated five-year bonds.

The AAA ratings for Mezz ABS CDOs are much less defensible. Scenarios where all the underlying BBB tranches lose virtually all their principal are sufficiently probable that it is not reasonable to assign a AAA rating to even a quite thin senior tranche. The risks in Mezz ABS CDOs depend critically on a) the correlation between pools, b) the tail default correlation, and c) the relationship between the recovery rate and the default rate. The very thin BBB tranches that were used accentuated the risks, but making the tranches wider would not have made the AAA ratings defensible. An important point is that the BBB tranche of an ABS cannot be assumed to be similar to a BBB bond for the purposes of determining the risks in ABS CDO tranches.

In practice, Mezz ABS CDOs accounted for about 3% of all mortgage securitizations, but they were a more prominent feature of financial markets than this statistic indicates. The AAA tranches of ABS CDOs were frequently used by market participants to create synthetic CDOs. Also the purchasers the tranches often bought protection against losses on them from third parties. The TABX index shows that ABS CDO tranches originally rated AAA had become worthless by mid-2009. An important implication of our research is that, when there is a rebirth of securitization, both regulators and market participants should very wary of resecuritizations (i.e., of any situation where tranches are formed from other tranches).

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Figure 1: Creation of Tranches from a Portfolio of Subprime Mortgages

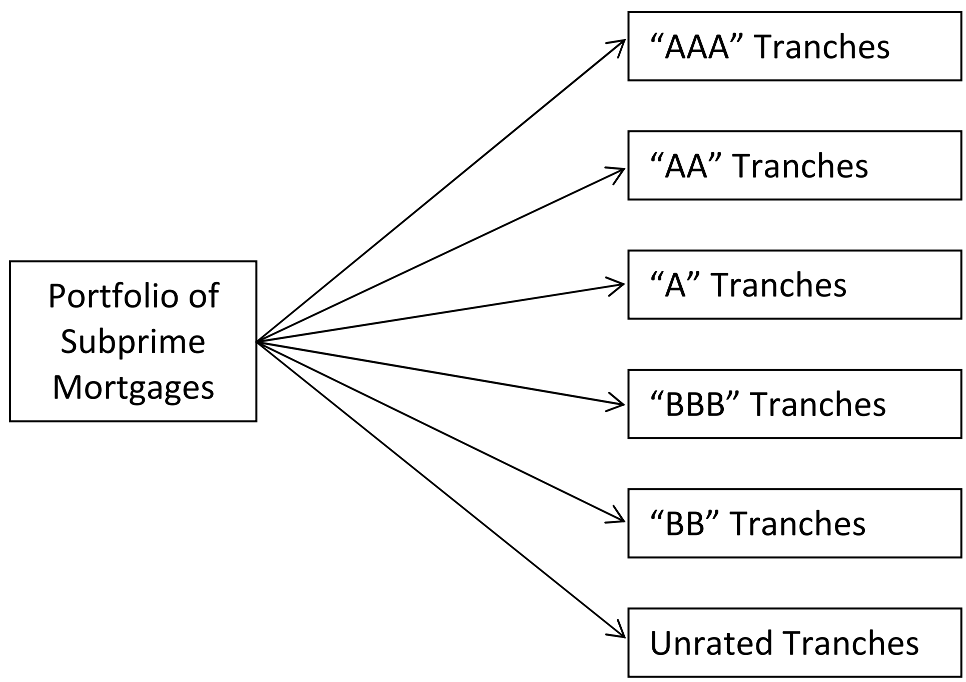


Figure 2: A "Mezz" ABS CDO Created from the BBB tranches of an ABS

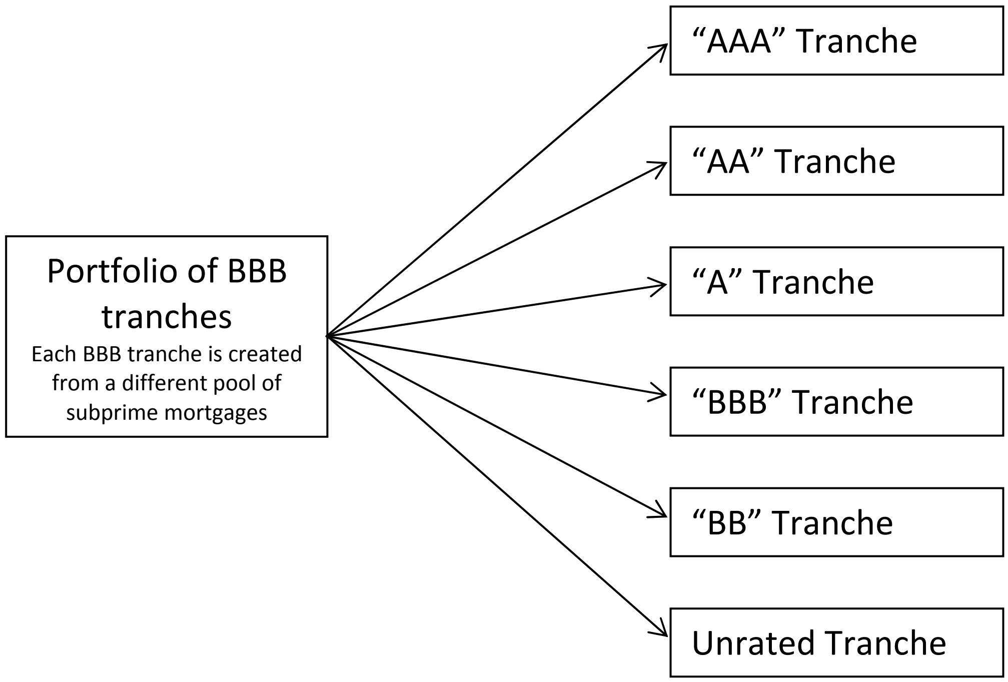


Figure 3: The Recovery Rate Model

The maximum recovery rate, $R_{\max}$, is 100%; the minimum recovery rate, $R_{\min}$, is 50%; the average recovery rate, R^*, is 75%; and the expected default rate is 10%.

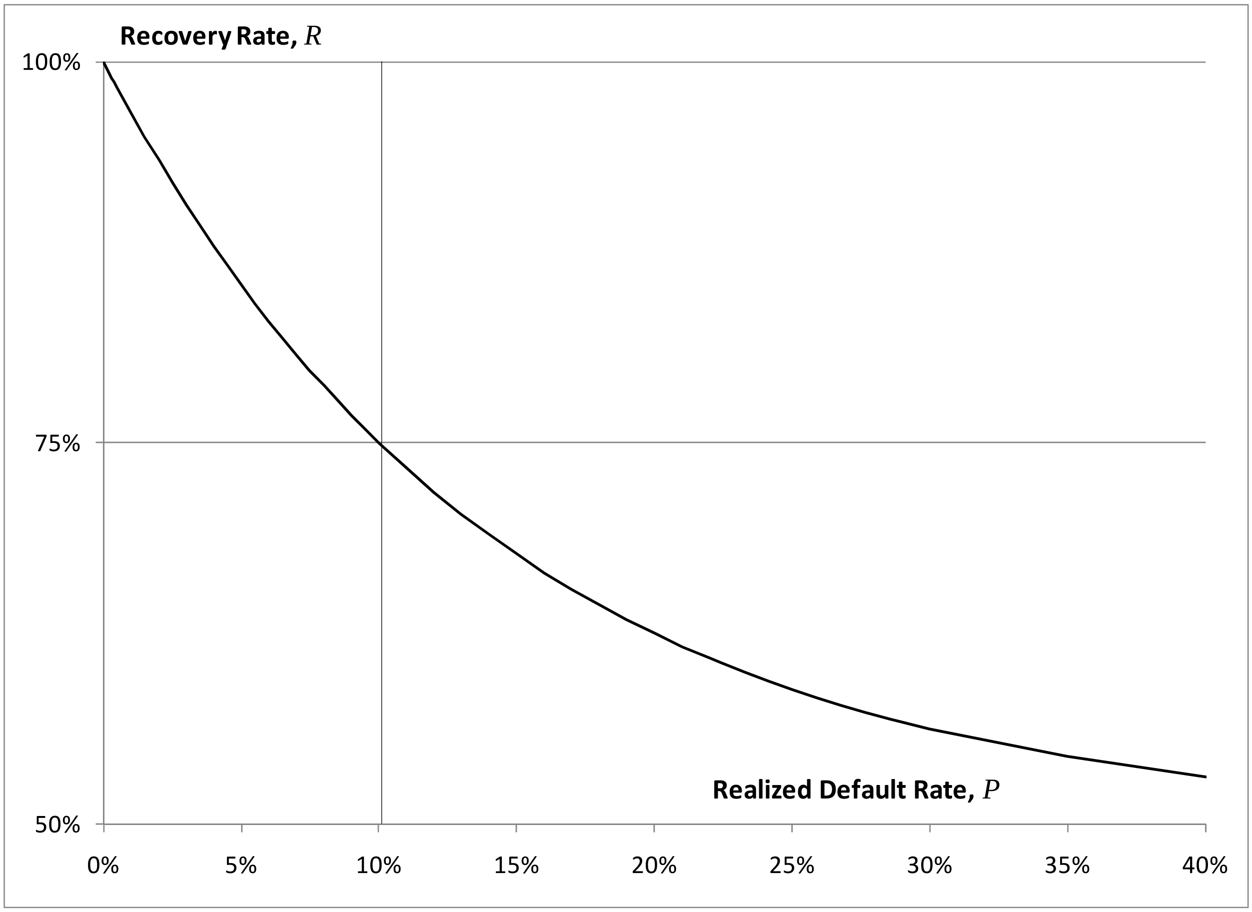
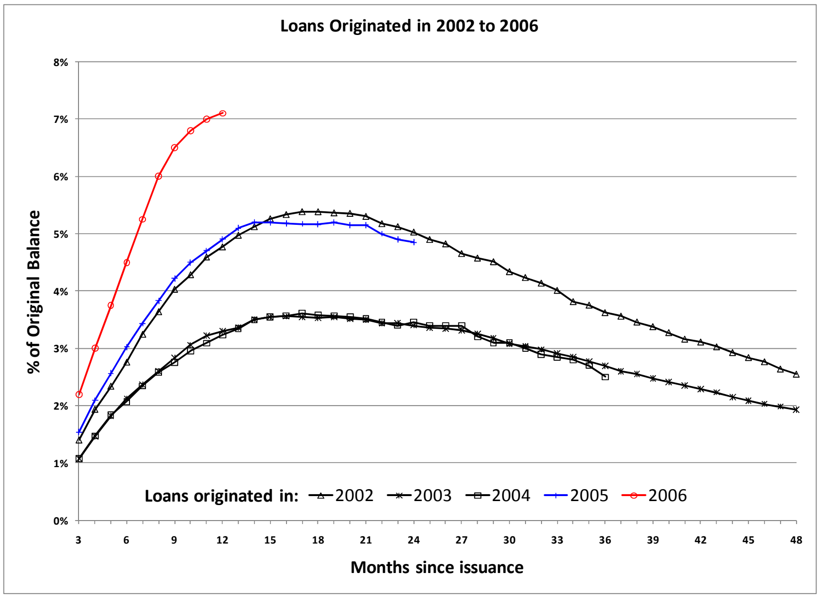


Figure 4: Data from March 2007 on Subprime Loans, 60 or More Days Delinquent, In Foreclosure or Held for Sale



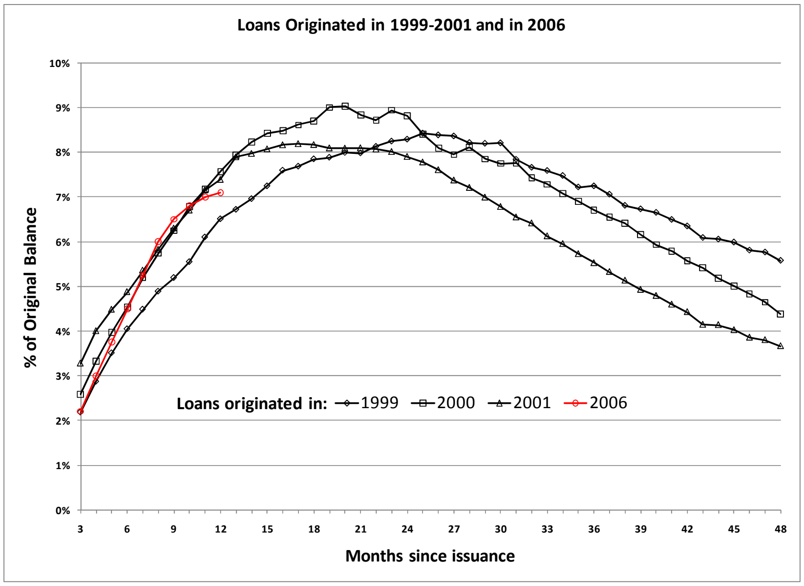


Table 1

Cumulative probability of default over 5 years for bonds initially rated AAA and BBB taken from Moody's statistics for the 1970 to 2007 period. Expected losses are calculated by assuming a recovery rate of 40%

	Probability of Loss	Expected Loss
AAA	0.1%	0.06%
BBB	1.8%	1.08%

Table 2

Minimum attachment point for the AAA rated tranche of an ABS to achieve a probability of loss less than 0.1%. In the constant recovery rate model the recovery rate is 75%. In the stochastic recovery rate model the recovery rate depends on the default rate and ranges from a high of 100% to a low of 50%.

		Expected Default Rate		
		5%	10%	20%
Gaussian Copula	$\rho = 0.05$	4.1%	6.8%	11.0%
Constant Recovery	$\rho = 0.10$	6.0%	9.4%	13.9%
	$\rho = 0.20$	9.6%	13.6%	18.2%
	$\rho = 0.30$	13.1%	17.2%	21.1%
Double *t* Copula	$\rho = 0.05$	7.6%	13.0%	18.2%
Constant Recovery	$\rho = 0.10$	13.6%	18.7%	21.9%
	$\rho = 0.20$	21.1%	23.2%	24.1%
	$\rho = 0.30$	23.7%	24.4%	24.7%
Gaussian Copula	$\rho = 0.05$	7.3%	11.6%	17.1%
Stochastic Recovery	$\rho = 0.10$	11.6%	17.3%	23.8%
	$\rho = 0.20$	19.1%	26.6%	33.4%
	$\rho = 0.30$	26.1%	34.1%	40.0%
Double *t* Copula	$\rho = 0.05$	15.0%	25.3%	33.4%
Stochastic Recovery	$\rho = 0.10$	27.2%	37.2%	41.8%
	$\rho = 0.20$	42.2%	46.3%	46.6%
	$\rho = 0.30$	47.4%	48.7%	47.8%

Table 3

Comparison of minimum attachment point for a AAA-rated tranche of an ABS when a) the expected loss criterion is used so that a AAA tranche is chosen to achieve an expected loss less than 0.06% and b) the probability of loss criterion is used so that a AAA tranche is chosen to achieve a probability of loss less than 0.1%.

The model is the double *t* copula model with a stochastic recovery rate. The recovery rate depends on the default rate and ranges from a high of 100% to a low of 50%.

		Expected Default Rate		
		5%	10%	20%
Expected Loss Criterion	$\rho = 0.05$	3.9%	10.9%	19.7%
	$\rho = 0.10$	10.5%	21.2%	28.9%
	$\rho = 0.20$	24.7%	33.2%	37.3%
	$\rho = 0.30$	33.4%	39.0%	41.1%
Probability of Loss Criterion	$\rho = 0.05$	15.0%	25.3%	33.4%
	$\rho = 0.10$	27.2%	37.2%	41.8%
	$\rho = 0.20$	42.2%	46.3%	46.6%
	$\rho = 0.30$	47.4%	48.7%	47.8%

Table 4

Minimum attachment points for the AAA senior tranche of an ABS CDO. The ABS CDO is created from 100 BBB tranches of ABS tranches. The attachment point for each BBB tranche is 4% and the detachment point is 5%. The model determining the actual default rate is given in Section 2. The parameters α and ρ are defined so that the between pool copula correlation is αρ and the within pool correlation is ρ. EDR is the expected default rate.

		α = 0.05	α = 0.25	α = 0.50	α = 0.75	α = 0.95
Gaussian Copula	ρ = 0.05	17.1%	42.7%	73.5%	96.2%	99.9%
Constant Recovery	ρ = 0.10	29.7%	62.3%	89.7%	99.8%	99.9%
EDR=10%	ρ = 0.20	39.7%	73.6%	95.4%	99.9%	99.9%
	ρ = 0.30	43.5%	77.2%	96.7%	99.9%	99.9%
Gaussian Copula	ρ = 0.05	0.9%	2.6%	5.9%	10.1%	10.4%
Constant Recovery	ρ = 0.10	5.3%	16.1%	36.2%	66.3%	98.3%
EDR=5%	ρ = 0.20	14.5%	37.9%	69.1%	95.2%	99.9%
	ρ = 0.30	20.5%	48.8%	80.2%	98.7%	99.9%
Triple *t* copula	ρ = 0.05	95.9%	100.0%	100.0%	100.0%	100.0%
Stochastic Recovery	ρ = 0.10	93.8%	100.0%	100.0%	100.0%	100.0%
EDR=10%	ρ = 0.20	92.0%	100.0%	100.0%	100.0%	100.0%
	ρ = 0.30	90.3%	100.0%	100.0%	100.0%	100.0%
Triple *t* Copula	ρ = 0.05	82.9%	99.0%	100.0%	100.0%	100.0%
Stochastic Recovery	ρ = 0.10	84.1%	99.0%	100.0%	100.0%	100.0%
EDR=5%	ρ = 0.20	85.0%	99.0%	100.0%	100.0%	100.0%
	ρ = 0.30	80.0%	99.0%	100.0%	100.0%	100.0%

Table 5

Minimum attachment points for the AAA senior tranche of an ABS CDO. The ABS CDO is created from 100 BBB tranches of ABS tranches. The attachment point for each BBB tranche is 4% and the detachment point is 7%. The model determining the actual default rate is given in Section 2. The parameters α and ρ are defined so that the between pool copula correlation is $\alpha\rho$ and the within pool correlation is ρ. EDR is the expected default rate.

		$\alpha = 0.05$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.95$
Gaussian Copula	$\rho = 0.05$	8.1%	22.5%	43.2%	66.2%	85.7%
Constant Recovery	$\rho = 0.10$	18.2%	43.7%	72.4%	94.2%	99.9%
EDR=10%	$\rho = 0.20$	29.6%	61.6%	88.5%	99.5%	99.9%
	$\rho = 0.30$	35.2%	68.5%	92.8%	99.9%	99.9%
Gaussian Copula	$\rho = 0.05$	0.0%	1.1%	2.2%	3.6%	3.5%
Constant Recovery	$\rho = 0.10$	2.7%	8.6%	19.9%	37.2%	58.6%
EDR=5%	$\rho = 0.20$	9.6%	27.1%	53.9%	83.9%	99.9%
	$\rho = 0.30$	15.2%	39.3%	70.0%	94.9%	99.9%
Triple *t* copula	$\rho = 0.05$	90.7%	99.6%	100.0%	100.0%	100.0%
Stochastic Recovery	$\rho = 0.10$	89.9%	99.7%	100.0%	100.0%	100.0%
EDR=10%	$\rho = 0.20$	88.7%	99.7%	100.0%	100.0%	100.0%
	$\rho = 0.30$	88.0%	99.4%	100.0%	100.0%	100.0%
Triple *t* Copula	$\rho = 0.05$	67.4%	97.4%	100.0%	100.0%	100.0%
Stochastic Recovery	$\rho = 0.10$	74.7%	98.4%	100.0%	100.0%	100.0%
EDR=5%	$\rho = 0.20$	76.7%	98.6%	100.0%	100.0%	100.0%
	$\rho = 0.30$	77.5%	99.0%	100.0%	100.0%	100.0%

Table 6

Minimum attachment points for the AAA senior tranche of an ABS CDO. The ABS CDO is created from 100 BBB tranches of ABS tranches. The attachment point for each BBB tranche is 4% and the detachment point is 9%. The model determining the actual default rate is given in Section 2. The parameters α and ρ are defined so that the between pool copula correlation is $\alpha\rho$ and the within pool correlation is ρ. EDR is the expected default rate.

		$\alpha = 0.05$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.95$
Gaussian Copula	$\rho = 0.05$	5.0%	13.9%	26.9%	41.2%	52.2%
Constant Recovery	$\rho = 0.10$	12.2%	30.7%	54.2%	77.0%	94.8%
EDR=10%	$\rho = 0.20$	22.6%	50.5%	78.8%	96.8%	99.9%
	$\rho = 0.30$	28.7%	60.1%	87.2%	99.3%	99.9%
Gaussian Copula	$\rho = 0.05$	0.0%	0.0%	1.3%	2.1%	2.1%
Constant Recovery	$\rho = 0.10$	1.7%	5.4%	12.6%	23.0%	35.2%
EDR=5%	$\rho = 0.20$	6.8%	19.9%	41.2%	68.0%	92.9%
	$\rho = 0.30$	11.7%	31.8%	59.9%	87.9%	99.9%
Triple *t* copula	$\rho = 0.05$	84.7%	98.8%	100.0%	100.0%	100.0%
Stochastic Recovery	$\rho = 0.10$	84.8%	99.4%	100.0%	100.0%	100.0%
EDR=10%	$\rho = 0.20$	85.2%	99.2%	100.0%	100.0%	100.0%
	$\rho = 0.30$	85.9%	99.0%	100.0%	100.0%	100.0%
Triple *t* Copula	$\rho = 0.05$	53.4%	95.6%	99.8%	100.0%	100.0%
Stochastic Recovery	$\rho = 0.10$	67.1%	97.8%	100.0%	100.0%	100.0%
EDR=5%	$\rho = 0.20$	71.9%	98.3%	100.0%	100.0%	100.0%
	$\rho = 0.30$	71.3%	98.0%	100.0%	100.0%	100.0%

Table 7

The effect on the minimum attachment point of moving from the situation where attachment point for each BBB tranche is 4% and each BBB tranche is 3% wide to the situation where the attachment point for each BBB tranche is drawn from a uniform distribution between 2% and 6% and the tranche width is drawn from a uniform distribution (independent of the first uniform distribution) between 1% and 5%. The model determining the actual default rate is given in Section 2. The parameters α and ρ are defined so that the between pool copula correlation is αρ and the within pool correlation is ρ. The expected default rate is 10%.

		α = 0.05	α = 0.25	α = 0.50	α = 0.75	α = 0.95
Gaussian Copula	ρ = 0.05	5.2%	4.9%	−2.8%	−1.7%	−12.2%
Constant Recovery	ρ = 0.10	3.1%	−0.7%	−3.4%	−5.4%	−2.3%
	ρ = 0.20	2.0%	3.9%	−2.4%	−2.2%	0.0%
	ρ = 0.30	0.2%	−0.2%	0.6%	−0.5%	0.0%
Triple *t* Copula	ρ = 0.05	−0.6%	−0.1%	0.0%	0.0%	0.0%
Stochastic Recovery	ρ = 0.10	3.5%	0.5%	0.0%	0.0%	0.0%
	ρ = 0.20	−1.4%	0.6%	0.0%	0.0%	0.0%
	ρ = 0.30	2.6%	0.0%	0.0%	0.0%	0.0%